Optimal Packings of Three Circles on Any Flat Torus

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1 Introduction

We determine the local and global maximally dense arrangements of three circles on any flat torus. We use a bigger circle and two smaller circles in our packing. Let C_b denote the big circle with radius r_b, let C_{s_1} denote one of the two small circles with radius r_s, and let C_{s_2} denote the other of the two small circles with radius r_s. Assume $\frac{r_s}{r_b} = \sqrt{2}-1$, chosen as a result of Heppes' [1] bound on the density of circle packings on the plane with size ratio $\sqrt{2}-1$.

Given a packing of C_b, C_{s_1} and C_{s_2} on a flat torus T, we lift the packing on T to the Euclidean plane. There exists a lattice Λ generated by the basis vectors $\{\vec{w_1}, \vec{w_2}\}$ such that $T \cong \mathbb{R}^2/\Lambda$, where lifts of one packing element in the plane differ by elements of Λ. Also assume that $\vec{w_1} = \langle 1, 0\rangle$ and $\vec{w_2} = \langle L\cos\alpha, L\sin\alpha\rangle$ with $0 \leqslant L\cos\alpha \leqslant \frac{1}{2}$ and $\frac{\pi}{3} \leqslant \alpha \leqslant \frac{\pi}{2}$, as any torus is equivalent to one with these restrictions. We will adhere to this notation throughout the following discussion.

2 Upper Bounds on the Density and the Radii

Proposition 2.1. *For a packing on any flat torus, T, with three circles of radii r_b, r_s, r_s with $\frac{r_s}{r_b} = \sqrt{2}-1$, the density of the packing is upper bounded by $\frac{(4-2\sqrt{2})\pi}{4}$.*

Proof. Suppose there exists a packing of three circles C_b, C_{s_1}, C_{s_2} on a flat torus T, with $\frac{r_s}{r_b} = \sqrt{2}-1$. We lift the packing on T to the Euclidean plane. A theorem of Heppes [1] about such a packing tells us that the density of the densest packing of circles in the Euclidean plane with size ratio of $\sqrt{2}-1$ is $\frac{(4-2\sqrt{2})\pi}{4} \approx 0.92015$. The density of a packing on any flat torus is equal to the density of that packing lifted to the plane. Therefore, the density of any packing on any flat torus is upper bounded by $\frac{(4-2\sqrt{2})\pi}{4}$. □

Next, we find the upper bound on the radii r_b and r_s.

Proposition 2.2. *For a packing on a torus T with three circles of radii r_b, r_s, r_s with $\frac{r_s}{r_b} = \sqrt{2}-1$, r_b is less than or equal to*

$$\min\{(\tfrac{1}{2}\sqrt{\tfrac{(-3+2\sqrt{2})L\sin\alpha}{-7+4\sqrt{2}}}), \tfrac{1}{2}\}$$

and r_s is less than or equal to

$$\min\{(\sqrt{2}-1)(\tfrac{1}{2}\sqrt{\tfrac{(-3+2\sqrt{2})L\sin\alpha}{-7+4\sqrt{2}}}), (\sqrt{2}-1)\tfrac{1}{2}\}.$$

Proof. We have that the density of a packing of C_b, C_{s_1}, C_{s_2} on a flat torus, $\frac{2\pi(r_s)^2+\pi(r_b)^2}{L\sin\alpha}$, is less than or equal to $\frac{4-2\sqrt{2}\pi}{4}$ by Proposition 2.1. We solve for r_b and we get that

$$r_b \leqslant \frac{1}{2}\sqrt{\frac{(-3+2\sqrt{2})L\sin\alpha}{-7+4\sqrt{2}}}.$$

Suppose, for the sake of a contradiction, that $r_b > \frac{1}{2}$. We may lift our packing to the Euclidean plane and using the trivial translations of the plane, we may assume that the center of C_b is located at $(0,0)$. Because $||\vec{w_1}|| = 1$, the centers of the closets lifts of C_b are one unit away from the center of C_b. By our assumption, $r_b > \frac{1}{2}$, so with a lift at $\vec{w_1}$, there is an overlapping of C_b with one of its closest lifts. This is a contradiction with the definition of a packing. To resolve this issue of overlapping, r_b must be less than or equal to $\frac{1}{2}$. We have the desired result.

Finally, we may now determine the upper bound on r_s. We have that $r_s = r_b(\sqrt{2}-1)$. Therefore, the maximum value of r_b on any flat torus also gives the maximum value of r_s on any flat torus. Thus we get that the maximum value of r_s is $\min\{(\sqrt{2}-1)(\frac{1}{2}\sqrt{\frac{(-3+2\sqrt{2})L\sin\alpha}{-7+4\sqrt{2}}}), (\sqrt{2}-1)\frac{1}{2}\}$. □

These two results will help us in determining the maximum number tangencies between our circles on any flat torus.

3 Maximum Tangencies Between C_b, C_{s_1}, and C_{s_2} on Any Flat Torus T, With $\frac{r_1}{r_2} = \sqrt{2}-1$

We start by defining what it means for a pair of circles to be $n-$tangent.

Defintion 3.1. *In a circle packing on a torus if a pair of circles share n different points of tangencies, we call them $n-$tangent.*

In the Euclidean plane, a pair of circles are at most 1-tangent to each other, and no circle is self tangent; however, this is not the case on a flat torus as the results of this section will show. We first consider how many tangencies there can be between C_{s_i}, for $i = 1, 2$ in our packing.

Proposition 3.2. *Consider a packing of three circles C_b, C_{s_1}, and C_{s_2} on any flat torus T, with $\frac{r_1}{r_2} = \sqrt{2}-1$. C_{s_1} and C_{s_2} can be at most 1-tangent to each other.*

Proof. We prove by contradiction. Suppose there exists a packing of three circles C_b, C_{s_1}, C_{s_2} on a flat torus T, with $\frac{r_s}{r_b} = \sqrt{2}-1$ where C_{s_1} and C_{s_2} are 2-tangent or more. Lift the packing on T to the Euclidean plane.

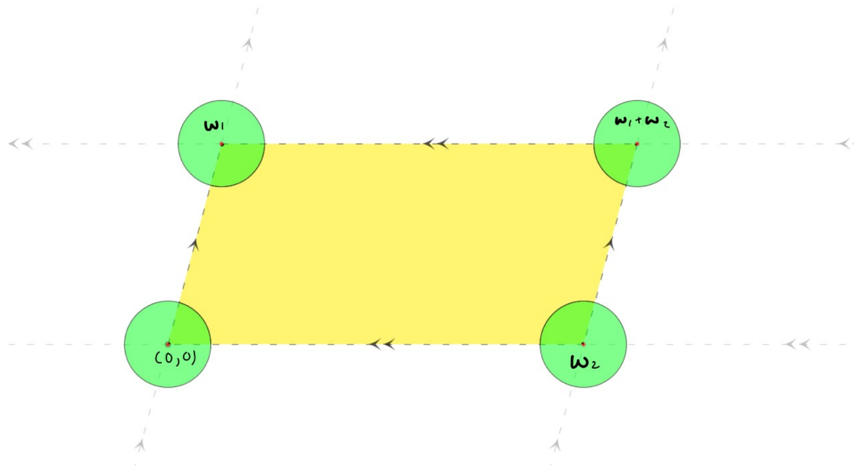


Figure 1: The lifts of C_{s_1}.

Using the trivial translations of the plane, let the center of C_{s_1} is located at $(0,0)$ and C_{s_2} is located in a fundamental domain bounded by $\vec{w_1}$ and $\vec{w_2}$. As C_{s_1} and C_{s_2} are 2-tangent the circle C_{s_2} must be tangent to two lifts of C_{s_1} located at the points $(0,0)$ and the end points of $\vec{w_1}$ or $\vec{w_2}$ or $\vec{w_1}+\vec{w_2}$. See Figure 1.

If C_{s_2} is tangent to lifts of C_{s_1} at $(0,0)$ and the end point of $\vec{w_1}$, notice that the center of C_{s_2} must be on the perpendicular bisector of the line segment from $(0,0)$ to $\vec{w_1}$. The smallest possible radius of C_{s_2} occurs when C_{s_2} is on this line segment. At this location the radius of r_s is required to be $\frac{1}{4}$ for a 2-tangency between C_{s_1} and C_{s_2}. Therefore the radius of both C_{s_1} and C_{s_2} must be at least $\frac{1}{4}$. However, by Proposition 2.2, the maximum value of r_s is

$$\min\{(\sqrt{2}-1)(\tfrac{1}{2}\sqrt{\tfrac{(-3+2\sqrt{2})L\sin\alpha}{-7+4\sqrt{2}}}),(\sqrt{2}-1)\tfrac{1}{2}\}\leqslant(\sqrt{2}-1)\tfrac{1}{2}<\tfrac{1}{4}.$$

If C_{s_2} is tangent to the lifts of C_{s_1} at $(0,0)$ and the endpoints of $\vec{w_2}$ or $\vec{w_1}+\vec{w_2}$, or any other lattice vector, we can repeat this argument to get a contradiction as for all $\vec{v}$ in Λ, $\vec{v}\geqslant 1$. Therefore, C_{s_1} and C_{s_2} cannot be 2-tangent. □

Remark. *For the existence of a 1-tangency between C_{s_1} and C_{s_2}, consider the case when $||\vec{w_2}||=1$ and the end point of $\vec{w_2}$ is located at $(0,1)$. We now use the trivial translations of the plane to assume that the center of C_b is located at $(0,0)$ and C_{s_1} and C_{s_2} are located in the fundamental domain bounded by $\vec{w_1}$ and $\vec{w_2}$. There exists a lift of C_b whose center is located at $(1,0)$. We move the centers of C_{s_1} and C_{s_2} to the line segment from $(0,0)$ to $(1,0)$ as shown in Figure 2.*

This is possible when $r_b=\frac{1}{2+4(\sqrt{2}-1)}$. This packing gives us a 1-tangency between C_{s_1} and C_{s_2}. Thus, C_{s_1} and C_{s_2} can be at most 1-tangent to each other.

Now we show that C_{s_i}, for $i=1,2$, cannot be self tangent on a flat torus. The essence of this argument is the same as the previous one, in which we showed that C_{s_i}, for $i=1,2$, cannot be 2-tangent on a flat torus.

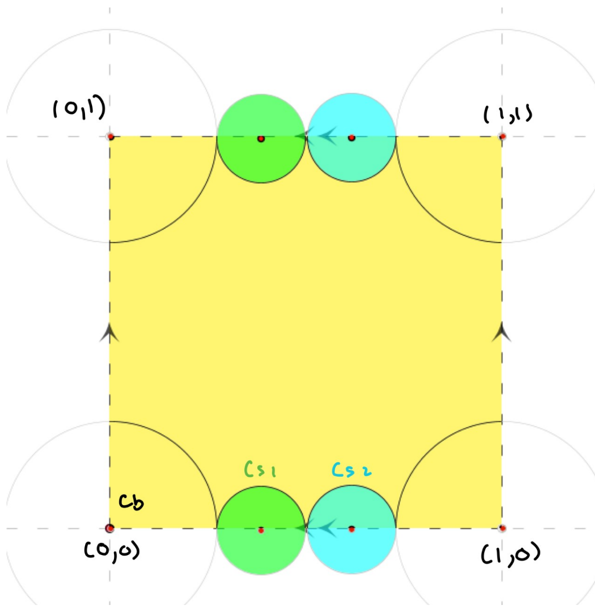


Figure 2: This packing gives us a 1-tangency between C_{s_1} and C_{s_2}.

Proposition 3.3. *Consider a packing of three circles C_b, C_{s_1}, and C_{s_2} on any flat torus T, with $\frac{r_1}{r_2} = \sqrt{2}-1$. Neither C_{s_1} nor C_{s_2} can be self-tangent.*

Proof. We prove by contradiction. Suppose there exists a packing of three circles C_b, C_{s_1}, C_{s_2} on a flat torus T, with $\frac{r_s}{r_b} = \sqrt{2}-1$ where, without loss of generality, C_{s_1} is self-tangent. We lift the packing on T to the Euclidean plane.

Using the trivial translations of the plane, we may assume, without loss of generality, that the center of C_{s_1} is located at $(0,0)$. We know from Proposition 2.2, the maximum value of r_s is less than or equal to $(\sqrt{2}-1)\frac{1}{2}$. Since C_{s_1} is self-tangent, we have that the distance between the center of C_{s_1} and at least one of its lifts, must be less than or equal to $(\sqrt{2}-1)$. The distance between the center of C_{s_1} and its lift with the center located at the endpoint of $\vec{w_1}$ is the shortest (and so if another lift is used, the problem illustrated here is worse), as for all $\vec{v}$ in Λ, $\vec{v} \geqslant 1$. If C_{s_1} is self-tangent to itself along $\vec{w_1}$, then $r_s = \frac{1}{2}$; however, $(\sqrt{2}-1) < \frac{1}{2}$. Therefore C_{s_1} is not self-tangent along $\vec{w_1}$ and we have a contradiction. We have showed that neither C_{s_1} nor C_{s_2} can be self-tangent. □

While we have that C_{s_i}, for $i = 1, 2$, cannot be self-tangent on a flat torus, the same is not true for C_b as the following proposition shows.

Proposition 3.4. *Consider a packing of three circles C_b, C_{s_1}, and C_{s_2} on any flat torus T, with $\frac{r_s}{r_b} = \sqrt{2}-1$. C_b is self-tangent at most once.*

Proof. We prove by contradiction. Suppose there exists a packing of three circles C_b, C_{s_1}, C_{s_2} on a flat torus T, with $\frac{r_s}{r_b} = \sqrt{2}-1$ where C_b is self-tangent twice. We lift the packing on T to the Euclidean plane.

Using the trivial translations of the plane, the center of C_b is located at $(0,0)$. We know from Proposition 2.2, the maximum value of r_b is $\frac{1}{2}$. As C_b is self-tangent twice, the distance between the centers of C_b and two of its lifts

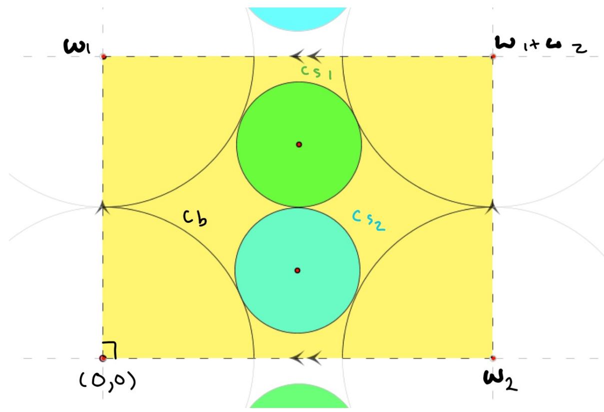


Figure 3: We have a self tangency of C_b along $\vec{w_1}$.

located at the endpoints of $\vec{w_1}$ or $\vec{w_2}$ or $\vec{w_1}+\vec{w_2}$ is less than or equal to 1. Note $||\vec{w_1}+\vec{w_2}|| > 1$, since $||\vec{w_1}|| = 1$, $||\vec{w_2}|| \geqslant 1$ with $\vec{w_2} = \langle L\cos\alpha, L\sin\alpha\rangle$ with $x^2+y^2 \geqslant 1$. That is C_b with the maximum value of $r_b = \frac{1}{2}$ can not be self-tangent along $\vec{w_1}+\vec{w_2}$. Using a similar argument we can also show that C_b can not be self-tangent along $-\vec{w_1}+\vec{w_2}, -\vec{w_1}-\vec{w_2}, \vec{w_1}-\vec{w_2}$. Since all other lifts of C_b located at $n\vec{w_1}+m\vec{w_2}$ where $|n|$ and $|m| > 1$ is farther from the origin, we needn't consider them. So, the lifts of C_b that will give us the two distinct points of self tangency of C_b will have their centers at the endpoints of $\vec{w_1}$ (and therefore also $-\vec{w}_1$, but these tangencies are equivalent and do not count twice) and $\vec{w_2}$ (and therefore also $-\vec{w}_2$, but these tangencies are equivalent and do not count twice).

To achieve a self tangency of C_b along $\vec{w_1}$, we need $r_b = \frac{1}{2}$ as $||\vec{w_1}|| = 1$ and $2(r_b) = 2(\frac{1}{2}) = 1$. This implies that the distance from $(0,0)$ to the center of the lift at w_2 is also 1, and so $L = 1$. This forces $r_s = (\sqrt{2}-1)\frac{1}{2}$. We calculate the density, d, of this packing in terms of α, the angle between $\vec{w_1}$ and $\vec{w_2}$, and get that

$$d = \frac{\pi(\frac{1}{2})^2 + 2\pi(\frac{1}{2}(\sqrt{2}-1))^2}{\sin\alpha}.$$

As $\frac{\pi}{3} \leqslant \alpha \leqslant \frac{\pi}{2}$, the minimum value of d is $\pi(\frac{1}{2})^2 + 2\pi(\frac{1}{2}(\sqrt{2}-1))^2 \approx 1.05$ as the function $\sin(x)$ from $\mathbb{R}$ to $\mathbb{R}$ is increasing on $[0, \frac{\pi}{2}]$.

This density violates Proposition 2.1, so we have a contradiction and C_b does not have two distinct points of self tangency.

Thus, C_b can have at most one point of self tangency. □

Remark. *For the existence of a self tangency, consider the packing of C_b, C_{s_1}, C_{s_2} on a $1\times L$ ($L \approx 1.29$) rectangular torus where $L \geqslant 1$. Then we have a self tangency of C_b along $\vec{w_1}$. See Figure 3.*

Now that we have established the fact that C_b can be self-tangent, we show that this is only possible if and only if $r_b = 1/2$. We also show that a self-tangency of C_b implies that the magnitude of our second vector, $\vec{w_2}$ is greater than 1.

Proposition 3.5. *Consider a packing of three circles C_b, C_{s_1}, C_{s_2} on a flat torus T with $\frac{r_s}{r_b} = \sqrt{2} - 1$. C_b is self-tangent if and only if $r_b = \frac{1}{2}$. And when C_b is self-tangent, we have that $||\vec{w_2}| > 1|$.*

Proof. We prove this directly. We lift the packing on T to the Euclidean plane. First we show that if $r_b = \frac{1}{2}$, then C_b is self-tangent. Note that $||\vec{w_1}|| = 1$. As $r_b = \frac{1}{2}$, the lift of C_b at $(0,0)$ and the lift of C_b at the end point of $\vec{w_1}$ are tangent.

Now we show that if C_b is self-tangent then $r_b = \frac{1}{2}$. Let C_b be self-tangent, that is C_b centered at $(0,0)$ is tangent to at least one of the lifts of C_b centered at $\vec{w_1}$ or $\vec{w_2}$. This is because if C_b at $(0,0)$ is self-tangent and the other lift to which it is tangent is anywhere other than $\vec{w_1}$, then the radius would be bigger than 1/2 and we would have overlaps between the lifts. We have that $||\vec{w_1}|| \leqslant ||\vec{w_2}||$ and, in our case, $||\vec{w_1}|| < ||\vec{w_1} + \vec{w_2}||$. So if C_b, centered at $(0,0)$, was tangent to the lifts of C_b centered at $\vec{w_2}$ or $\vec{w_1} + \vec{w_2}$, then $r_b = \frac{1}{2}||\vec{w_2}||$ or $r_b = \frac{1}{2}||\vec{w_1} + \vec{w_2}||$ respectively. That is, $2r_b \geqslant 1$. For $2r_b = 1$, $||\vec{w_2}|| = 1$. However, we know from our argument in Proposition 3.4 that $||\vec{w_2}|| > 1$. Therefore, $2r_b > 1$; however, if $2r_b > 1$ then along $\vec{w_1}$, we have that the lifts of C_b overlap, since $||\vec{w_1}|| = 1$. Therefore, C_b must be self-tangent along $\vec{w_1}$. This gives us that $r_b = \frac{1}{2}$ as $||\vec{w_1}|| = 1$.

Thus, C_b is self-tangent if and only if $r_b = \frac{1}{2}$. □

We will be using the following proposition to prove some of our other propositions that follow.

Proposition 3.6. *Consider a packing of C_b, C_{s_1}, C_{s_2} on a flat torus T with $\frac{r_s}{r_b} = \sqrt{2} - 1$. If C_{s_i}, for $i = 1, 2$ and C_b are tangent then we can always lift the packing on T to the Euclidean plane where*

1. *C_b is centered at $(0,0)$*
2. *C_{s_i} is in the fundamental domain spanned by $\vec{w_1}$ and $\vec{w_2}$*
3. *C_{s_i} is tangent to the lifts of C_b centered at $(0,0), \vec{w_1}, \vec{w_2}$ or $\vec{w_1} + \vec{w_2}$.*
4. *If C_{s_i} is tangent to the lift of C_b centered at $(0,0)$, then it can not be tangent to the lift of C_b centered at the endpoint of $\vec{w_1} + \vec{w_2}$.*

Proof. We prove (3) by contradiction. Suppose that C_{s_i} was tangent to a lift of C_b centered at $n\vec{w_1} + m\vec{w_2}$ where (n,m) is not in the set $\{(0,0),(1,0),(0,1),(1,1)\}$. Then the distance between the centers of C_{s_1} and that lift of C_b must be less than or equal to the maximum value of r_b plus the maximum value of r_s, which is $\frac{1}{2} + \frac{1}{2}(\sqrt{2} - 1) = \frac{1}{2} \approx 0.707$. The shortest distance between a point in the fundamental domain spanned by $\vec{w_1}$ and $\vec{w_2}$ and a lift of C_b centered at $n\vec{w_1} + m\vec{w_2}$ where at least one of $|n|, |m| > 1$, is d_1 or d_2 as shown in Figure 4. This is because the shortest distance between two parallel lines is the length of the perpendicular segment between them and for a fixed basis, the edges of the fundamental domain forms parallel lines with their respective lifts with when lifted to the plane. That is, if we have a point in the fundamental domain spanned by $\vec{w_1}$ and $\vec{w_2}$ and one of the lifts of C_b centered at $n\vec{w_1} + m\vec{w_2}$, where at least one of $|n|, |m| > 1$, then the shortest distances between the point and the

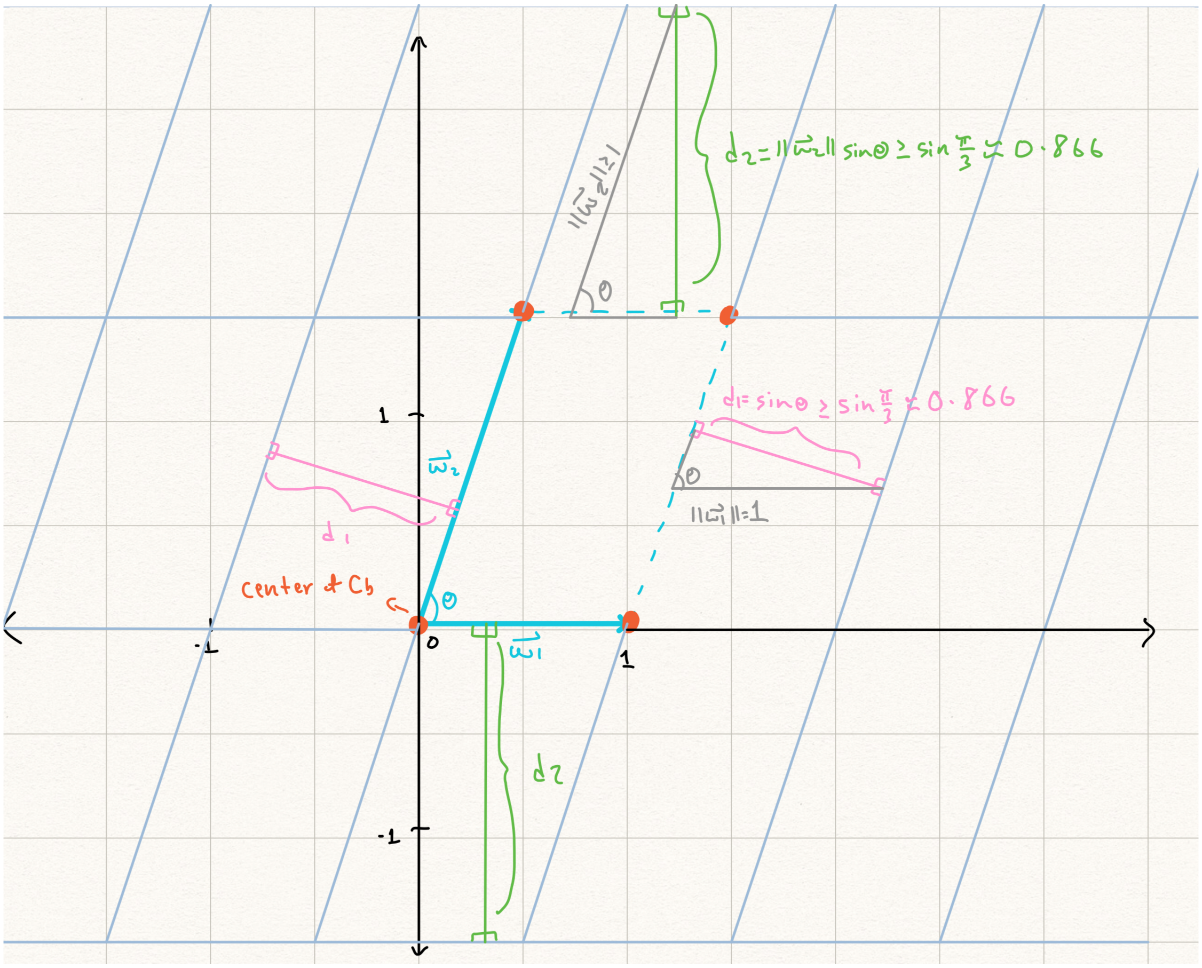


Figure 4: The shortest distance between a point in the fundamental domain spanned by $\vec{w_1}$ and $\vec{w_2}$ and a lift of C_b centered at $n\vec{w_1} + m\vec{w_2}$ where at least one of $|n|, |m| > 1$ is d_1 or d_2n

lift are the lengths of the perpendicular segment between the parallel lines: the edges of the fundamental domain and their respective lifts, d_1 and d_2.

Both d_1 and d_2 must be at least $\sin\frac{\pi}{3} \approx 0.866$. This value is based on the fact that $||\vec{w_2}|| \geqslant 1$, $\frac{\pi}{3} \leqslant \alpha \leqslant \frac{\pi}{2}$, and that $\sin x$ from $\mathbb{R}$ to $\mathbb{R}$ is increasing on $[\frac{\pi}{3}, \frac{\pi}{2}]$. This value, $\sin\frac{\pi}{3}$ is strictly greater than $\frac{1}{\sqrt{2}} \approx 0.707$, the value needed for a tangency of C_{s_1} with a lift of C_b centered at $n\vec{w_1}+m\vec{w_2}$ where (n, m) is not in the set $\{(0,0), (1,0), (0,1), (1,1)\}$. Therefore, we have a contradiction.

Now we prove (4) by contradiction. Suppose that C_{s_1} was tangent to the lifts of C_b centered at $(0,0)$ and the endpoint of $\vec{w_1} + \vec{w_2}$ simultaneously. Then the distance from $(0,0)$ to the endpoint of $\vec{w_1} + \vec{w_2}$ must be less than or equal to $\frac{1}{2} + (\sqrt{2} - 1) + \frac{1}{2} = \sqrt{2}$. This value is determined using the maximum values of r_b and r_s and the triangle inequality (the distance from $(0,0)$ to the endpoint of $\vec{w_1} + \vec{w_2}$ is the length of one side of the triangle with the other two sides having a length of $\frac{1}{2} + \frac{1}{2}(\sqrt{2} - 1)$).

The distance from $(0,0)$ to the endpoint of $\vec{w_1} + \vec{w_2}$ is

$$||\vec{w_1} + \vec{w_2}|| = \sqrt{(1 + L\cos(\alpha))^2 + (L\sin(\alpha))^2} = \sqrt{(1 + 2L\cos(\alpha) + L^2} \geqslant \sqrt{2},$$

because $L \geqslant 1$ and $\cos(\alpha)$.

The distance from $(0,0)$ to the endpoint of $\vec{w_1} + \vec{w_2}$ is equal to $\sqrt{2}$ only when $\alpha = \frac{\pi}{2}$ and $||\vec{w_2}|| = L = 1$. For all other values of α and L, $||\vec{w_1} + \vec{w_2}||$ is greater than $\sqrt{2}$. We calculate the density of this packing and get that it is

$$\frac{2\pi(\frac{1}{2})^2 + \pi(\frac{1}{2}(\sqrt{2}-1))^2}{1} \approx 1.7 > \frac{(4-2\sqrt{2})\pi}{4} \approx 0.92015,$$

which violates the results of Proposition 2.1, which gives the upper bound on the density of a packing with circles of radius ratio $\sqrt{2}-1$ on a flat torus. Therefore it not possible for C_{s_1} to be tangent to the lifts of C_b centered at $(0,0)$ and the endpoint of $\vec{w_1} + \vec{w_2}$ simultaneously. □

Now we can consider the tangencies between C_b and C_{s_i}, for $i = 1,2$, on a rectangular flat torus. We start with rectangular flat torus because the tangencies are more restricted on a rectangular flat torus than on a flat torus as the following two arguments will illustrate.

Defintion 3.7. *Let $R \cong \mathbb{R}^2/\Lambda$ where Λ is the lattice generated by the basis vectors $\{\vec{v_1}, \vec{v_2}\}$ such that $\vec{v_1} = \langle 1, 0\rangle$ and $\vec{v_2} = \langle 0, L\rangle$ with $L \geqslant 1$. We call R a $1 \times L$* rectangular flat torus.

Proposition 3.8. *Consider a packing of three circles C_b, C_{s_1}, C_{s_2} on a $1 \times L$ rectangular flat torus ($L \geqslant 1$) R, with $\frac{r_s}{r_b} = \sqrt{2}-1$. C_{s_i} and C_b are tangent at most twice, for $i = 1,2$.*

Proof. We prove by contradiction. Suppose there exists a packing of three circles C_b, C_{s_1}, C_{s_2} on rectangular flat torus T, with $\frac{r_s}{r_b} = \sqrt{2}-1$ where, without loss of generality, C_{s_1} is 3-tangent, or more to C_b. We lift the packing on T to the Euclidean plane. Since C_b is 3-tangent to C_{s_1}, we have, by Proposition 3.6, that the distance from C_{s_1} to the lifts of C_b at $(0,0)$, $\vec{w_1}$, and $\vec{w_2}$ is $r_s + r_b$.

To determine the location of the center of the lifts C_{s_1}, we can use the fact that the centers of C_b are located at the vertices of the fundamental domain of Λ, a rectangle, and at least three of these centers are equidistant from the center of C_{s_1}. Thus, C_{s_1} must be located where the diagonals of T intersect. See Figure 5.

By Proposition 2.2, $r_b \leqslant \frac{1}{2}$ and $r_s \leqslant \frac{1}{2}(\sqrt{2}-1)$. The upper bounds of r_b and r_s, independent of L, are $\frac{1}{2}$ and $\frac{1}{2}(\sqrt{2}-1)$, respectively. Since we are assuming a 3-tangency between C_{s_1} and C_b, we can use these upper bounds on the radii to find an upper bound on L that allows a 3-tangency between C_{s_1} and C_b. Refer to Figure 6 for the following results which rely on basic geometric properties.

We use the properties of right triangles to get that

$$\cos\alpha \geqslant \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{2}(\sqrt{2}-1)} = \frac{1}{\sqrt{2}},$$

and so

$$\alpha \leqslant \frac{\pi}{4}.$$

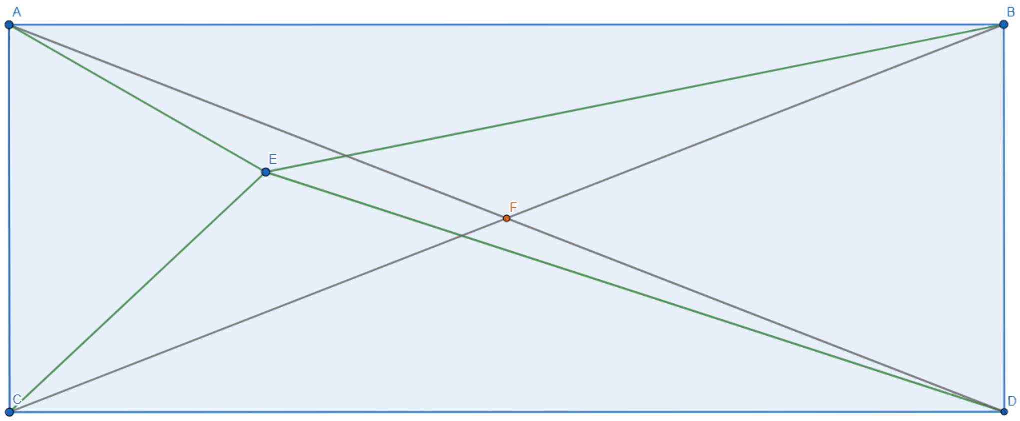


Figure 5: We want the point E to be equidistant from the points A, B and C and stay inside the blue rectangle. That is only possible if the point E is located at point F. Notice that point F is where the diagonal of the blue rectangle intersect.

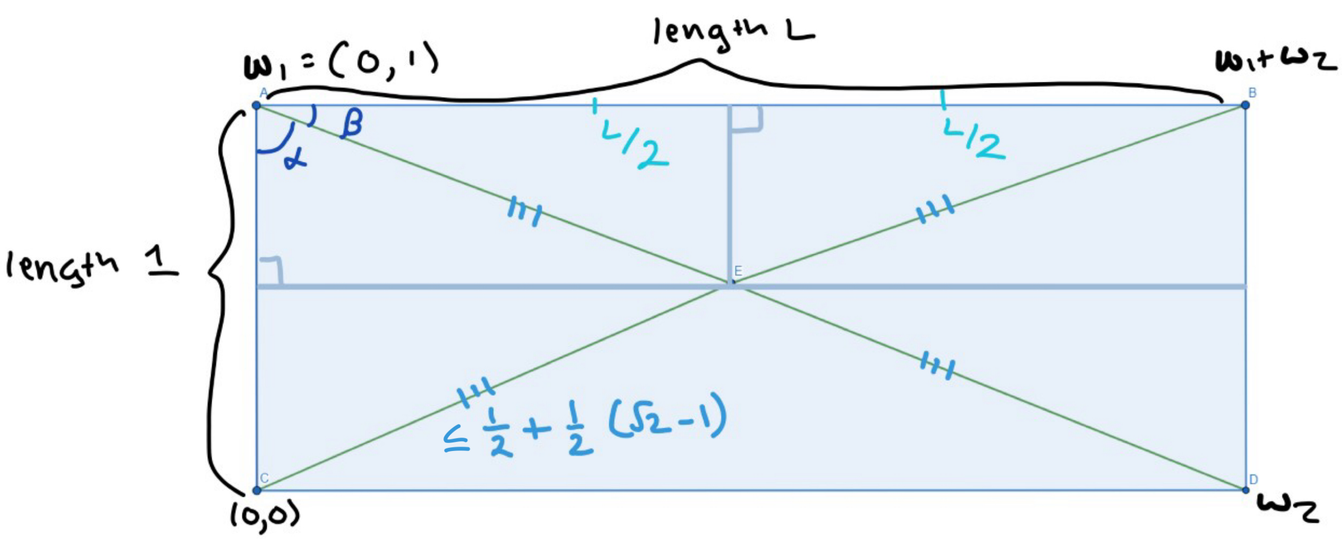


Figure 6: This figure describes the notation used in the proof of Proposition 3.8

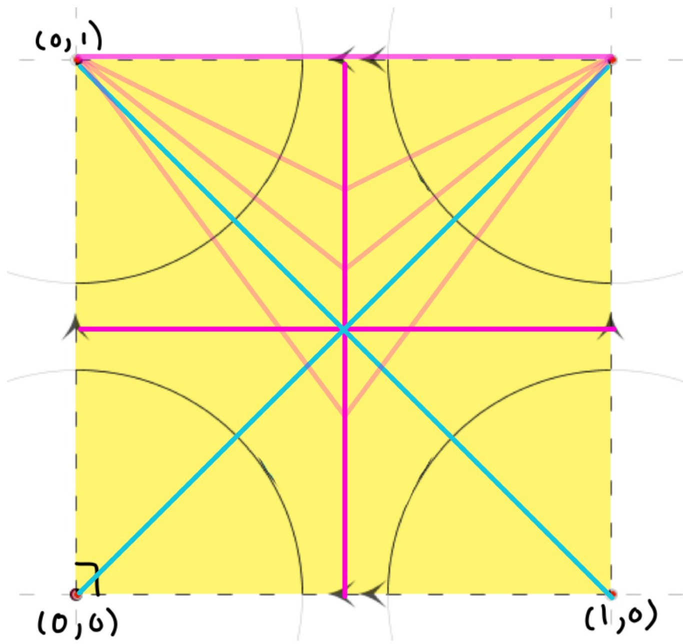


Figure 7: To achieve a 3-tangency between C_{s_1} and C_b, the center of C_{s_1} must be located where the perpendicular bisectors of $\vec{w_1}$ and $\vec{w_2}$ interect, as the center of C_{s_1} must be located along the perpendicular bisectors of the lines from $(0,0)$ to the endpoints of $\vec{w_1}$ and $\vec{w_2}$.

Using complementary angles, we get that $\alpha + \beta = \frac{\pi}{2}$ which implies $\beta \geqslant \frac{\pi}{4}$. So we have that $\cos\beta \leqslant \frac{1}{\sqrt{2}}$. Now we evaluate the cosine of β using our figure:

$$\cos\beta \geqslant \frac{\frac{L}{2}}{\frac{1}{2} + \frac{1}{2}(\sqrt{2}-1)} = \frac{L}{\sqrt{2}}.$$

We have that $\cos\beta \leqslant \frac{1}{\sqrt{2}}$ and that $\cos\beta \geqslant \frac{L}{\sqrt{2}}$ and that $L \geqslant 1$; thus the upper bound of L that allows a 3-tangency between C_b and C_{s_1} is when $L = 1$. However, since the lower and upper bounds on L are both 1, $L = 1$. When $L = 1$, and $\alpha = \pi/2$, Proposition 2.2, r_b is less than or equal to $\sqrt{\frac{(4-2\sqrt{2})\pi}{4(2\pi(\sqrt{2}-1)^2+\pi)}} \approx 0.47$ and r_s is less than or equal to $(\sqrt{2}-1)(\sqrt{\frac{(4-2\sqrt{2})}{4(2\pi(\sqrt{2}-1)^2+\pi)}}) \approx 0.19$.

To achieve a 3-tangency between C_{s_1} and C_b, the center of C_{s_1} must be located where the perpendicular bisectors of the lines from $(0,0)$ to the endpoints of $\vec{v_1}$ and $\vec{v_2}$ intersect, as the center of C_{s_1} must be located along the perpendicular bisectors of $\vec{v_1}$ and $\vec{v_2}$. See Figure 7.

This is exactly where the diagonals of our 1×1 square intersect. However, the length of this intersection point to the vertices of the square is $\frac{\sqrt{2}}{2} \approx 0.707 > 0.47 + 0.19$ and we have that C_b and C_{s_1} cannot be 3−tangent. This is a contradiction to our assumption.

Thus, there exists a maximum of two distinct tangencies between C_{s_1} and C_b, as well as between C_{s_2} and C_b on a rectangular flat torus. □

Remark. *For the existence of a 2-tangency between C_{s_i} and C_b, for $i = 1, 2$, refer to the results of Manning and*

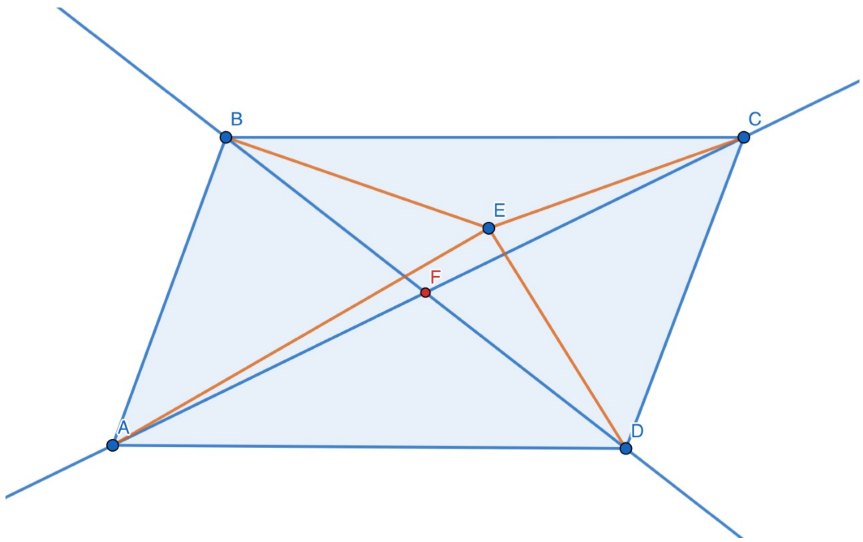


Figure 8: We want the point E to be equidistant from the points A, B and C and stay inside the blue rectangle. That is only possible if the point E is located at point F. Notice that point F is where the diagonal of the blue parallelogram intersect.

Parker [2] on the optimal packings of three circles with radii r_b, r_s, r_s, $\frac{r_s}{r_b} = \sqrt{2}-1$ on a square flat torus.

The previous argument only works for rectangular flat tori and not all tori, as the following proposition shows.

Proposition 3.9. *Consider a packing of three circles C_b, C_{s_1}, C_{s_2} on any flat torus T, with $\frac{r_s}{r_b} = \sqrt{2}-1$. C_{s_i} and C_b are tangent at most thrice, for $i = 1, 2$.*

Proof. We prove by contradiction. Suppose there exists a packing of three circles C_b, C_{s_1}, C_{s_2} on a flat torus T, with $\frac{r_s}{r_b} = \sqrt{2}-1$ where, without loss of generality, C_{s_1} is 4-tangent or more to C_b. We lift the packing on T to the Euclidean plane. Since C_b is 4-tangent to C_{s_1}, we have that the distance from C_b and all of its lifts, located at $(0,0)$ and the endpoints of $\vec{w_1}$ and $\vec{w_2}$ and $\vec{w_1} + \vec{w_2}$, is $r_b + r_s$. By Proposition 3.6, the other lifts of C_b needn't be considered.

Since the center of all of the lifts of C_b are located at the vertices of T, and they are all equidistant from the center of C_{s_1},we have that the center of C_{s_1} is located exactly where the diagonals of T intersect. See Figure 8.

Because the diagonals of a parallelogram are congruent if and only if the parallelogram is a rectangle, and the distance from the center of C_{s_1}, located where the diagonals of T intersect, to the centers of all the lifts of C_b located at the vertices of T is the same, T must be a rectangle. We know from Proposition 3.8 that on a rectangular torus, C_{s_1} and C_b cannot be more than 2-tangent; therefore our assumption that C_{s_1} and C_b were 4-tangent was wrong.

□

Remark. *For the existence of a 3-tangency between C_{s_i} and C_b, for $i = 1, 2$, refer to the results of Manning and Parker [2] on the optimal packings of three circles with radii r_b, r_s, r_s, $\frac{r_s}{r_b} = \sqrt{2}-1$ on a triangular flat torus.*

We have now determined the limits on the multiple tangencies between our circles on a flat torus. But we can

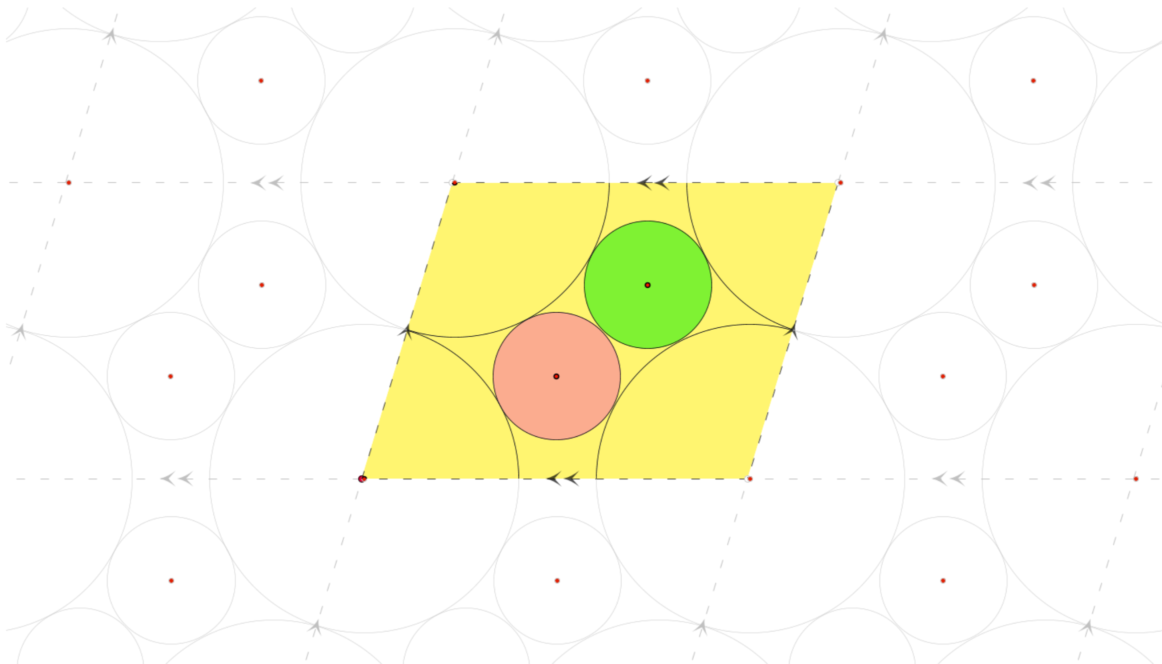


Figure 9: This is what a packing in which the conditions of Proposition 3.10 hold must look like. In this proof we assume that C_{s_1} is the pink circle and that C_{s_2} is the green circle

.

in fact improve the results of Proposition 3.9 by determining exactly which flat tori allow a 3-tangency between C_b and C_{s_i}, for $i = 1, 2$, and other conditions.

Proposition 3.10. *Consider a packing of three circles C_b, C_{s_1}, C_{s_1} on a flat torus T with $\frac{r_s}{r_b} = \sqrt{2} - 1$. Let α be the angle between $\vec{w_1}$ and $\vec{w_2}$. There exists a unique (α, L) pair such that all of the following are true: C_b is self-tangent, C_b is 3-tangent to both C_{s_1} and C_{s_2}, and C_{s_1} is tangent to C_{s_2}.*

Proof. We claim that such a packing must look like the packing depicted in Figure 9.

We lift the packing on T to the Euclidean plane. For C_b to be 3-tangent to C_{s_1}, three of these lifts must have their centers a distance of $r_b + r_s$ from C_{s_1}'s center. By Proposition 3.6, we may assume that these lifts of C_b are centered at $(0, 0)$, $\vec{w_1}$ and $\vec{w_2}$. Since the distances between the points $(0, 0)$ and the endpoints of $\vec{w_1}$ and $\vec{w_2}$ to the center of C_{s_1} are equal, the center of C_{s_1} must lie on the intersection of the perpendicular bisectors of the line segments from $(0, 0)$ to the endpoints of vectors $\vec{w_1}$ and $\vec{w_2}$. This means that C_{s_1} is centered on the circumcenter of the triangle with vertices $(0, 0)$, the endpoint of $\vec{w_1}$ and the endpoint of $\vec{w_2}$. See Figure 9.

Using a similar argument for C_{s_2}, if C_b is 3-tangent to C_{s_2} as well, then C_{s_2} is centered on the circumcenter of the triangle with vertices the endpoint of $\vec{w_1} + \vec{w_2}$, the endpoint of $\vec{w_1}$ and the endpoint of $\vec{w_2}$. Now, we force a tangency between the two smaller circles and we get the packing depicted in Figure 9.

We can now analyze the geometry of such a packing, as done in Figure 10. Most of the geometry in Figure 10 comes from the tangencies of the circles. We assign the values of β and θ to the angles between $\overline{RO}$ and $\overline{OP}$, and between $\overline{OP}$ and $\overline{OQ}$, respectively. Note that the angle β at the top left of Figure 10, in triangle OPR, is equal to

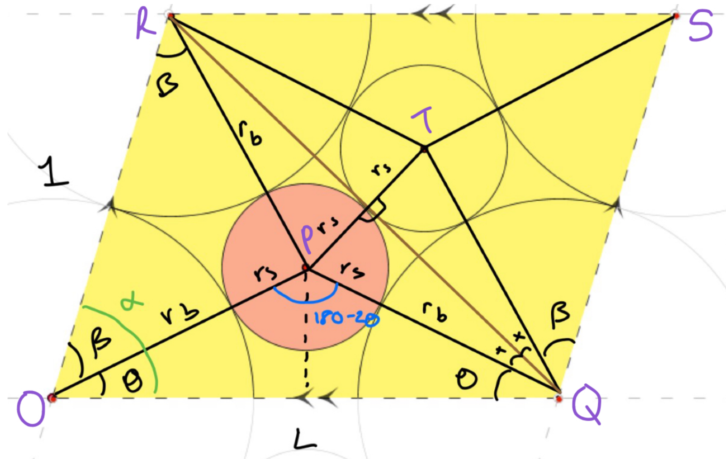


Figure 10: Refer to this Figure for the notation used in Proposition 3.10.

the angle β in the bottom right of Figure 10, in triangle QTS because the triangles that contain those angles are congruent.

The following notation refers to the notation in Figure 10. We first determine the value of α. To find α, we begin by solving for the angle x. We get the following equations using the properties of right triangles (the ones contained in rhombus $QPRT$):

$$\sin x = \frac{r_s}{r_b + r_s}, \tag{1}$$

and

$$\tan x = \frac{r_s}{\sqrt{(r_b + r_s)^2 + (r_s)^2}}. \tag{2}$$

We solve for x and get that $x = \arcsin(\frac{1}{2}(2 - \sqrt{2}))$. Then, using our value for x, as well as the equations $\pi = 2(\theta + \beta) + 2x$ (which we derive from the fact that $2x + \theta + \beta$ and $\alpha = \theta + \beta$ are supplementary angles) and $\alpha = \theta + \beta$, we get:

$$\alpha = \frac{1}{2}(\pi - 2\arcsin(1 - \frac{1}{\sqrt{2}})) \approx 1.27.$$

To find α and L we use the fact that C_b is tangent to itself and therefore has radius $\frac{1}{2}$. This also gives us the radius of C_{s_1} and C_{s_2} to be $\frac{1}{2}(\sqrt{2} - 1)$. To solve for α, we first find θ and β using properties of right triangles in terms of L. See Figure 11. We get that

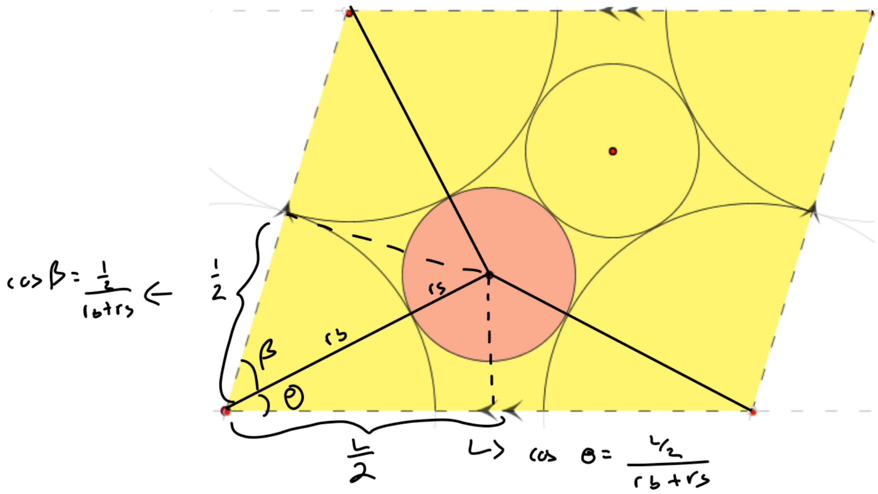


Figure 11: The derivations of equations (3) and (4) in Proposition 3.10 are shown in this figure.

$$\cos\theta = \frac{(\frac{1}{2})L}{r_s + r_b} = \frac{(\frac{1}{2})L}{\frac{1}{\sqrt{2}}}$$

and

$$\cos\beta = \frac{\frac{1}{2}}{r_s + r_b} = \frac{\frac{1}{2}}{\frac{1}{\sqrt{2}}}.$$

Now we use the fact that $\alpha = \theta + \beta$ to solve for α. We get that

$$\alpha = \arccos(\frac{(\frac{1}{2})L}{\frac{1}{\sqrt{2}}}) + \arccos(\frac{\frac{1}{2}}{\frac{1}{\sqrt{2}}}).$$

Now we solve for L to get $L = \sqrt{2}(\sin(\frac{\pi}{4} + \arcsin(1 - \frac{1}{\sqrt{2}}))) \approx 1.249$.

We have thus found the unique (α, L) pair that gives us self tangency of C_b, 3-tangency of C_b to both C_{s_1} and C_{s_2}, and tangency of C_{s_1} to C_{s_2}. □

4 The Associated Combinatorial Multi-Graphs

We have found the maximum number of tangencies and self tangencies between the circles C_b, C_{s_1} and C_{s_2} when packed on a flat torus. In this section we describe the possible combinatorial multi-graphs that can correspond to locally maximally dense packing with no free circless of C_b, C_{s_1} and C_{s_2} on a flat torus. The vertices in our combinatorial multi-graphs will represent the circles C_b, C_{s_1} and C_{s_2}. An edge between two vertices will denote a

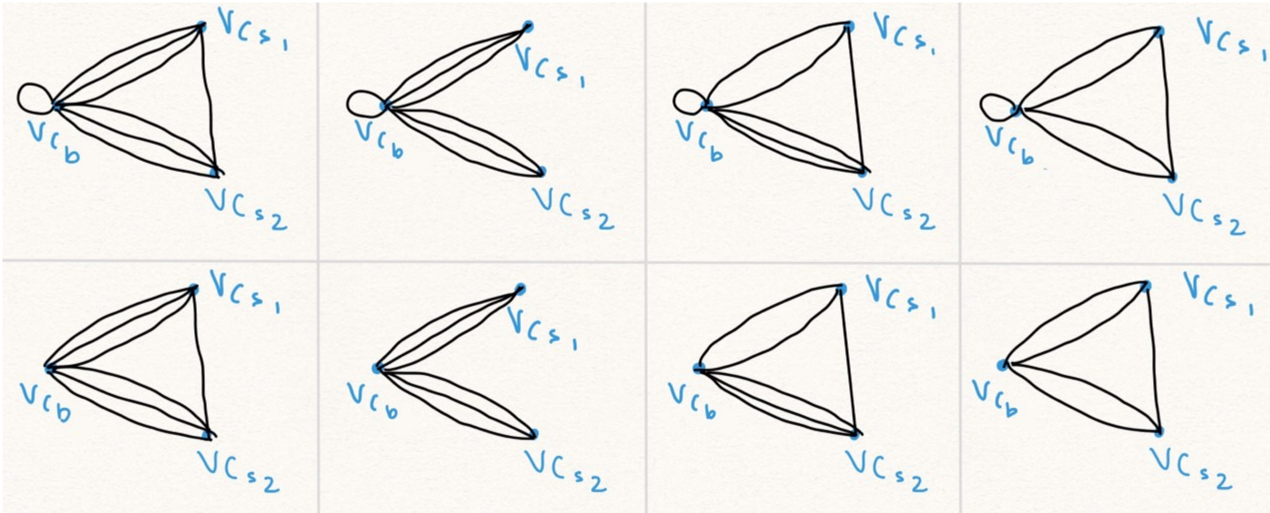


Figure 12: The possible combinatorial multi-graphs we can have that may correspond to locally maximally dense packing with no free circless of C_b, C_{s_1} and C_{s_2} on a flat torus with no free circles. v_{C_b} corresponds to C_b,and $v_{C_{s_i}}$ corresponds to C_{s_i} for $i = 1, 2$.

distinct tangency between the two circles that correspond to those two vertices. A loop on a vertex will denote a distinct self tangency of the circle that corresponds to that vertex. See Figure 12 for the possible combinatorial multi-graphs that might be associated to a locally maximally dense packing with no free circles.

Proposition 4.1. *The eight combinatorial multi-graphs in Figure 12 are all the possible combinatorial multi-graphs that can be associated to a locally maximally dense packing with no free circles of C_b, C_{s_1}, C_{s_2} on a flat torus T with $\frac{r_s}{r_b} = \sqrt{2} - 1$.*

Proof. In Section 3, we found the maximum number of tangenices between two circles in our packing as well as the maximum number of self tangenices of a circle in our packing. By Brandt, Dickinson, Ellsworth, Kenkel, and Smith [1] [3], every circle is tangent to at least three circles in a locally maximally dense packing with no free circles of circles on a flat torus with no free circles; therefore, each vertex on a combinatoral multi-graph must have at least degree three. Also, by Connelly [4], if $\mathcal{P}$ is a locally maximally dense packing with no free circles of n circles on a flat torus with no free circles, then the packing graph associated to $\mathcal{P}$ contains at least $2n - 1$ edges; since we have three circles, all combinatorial graphs representing locally maximally dense packing with no free circless with no free cricles must have at least five edges.

Let V_{C_b} be the vertex in our combinatorial multi-graph that corresponds to the C_b in our packing, let $V_{C_{s_1}}$ be the vertex in our combinatorial multi-graph that corresponds to the C_{s_1} in our packing, and let $V_{C_{s_2}}$ be the vertex in our combinatorial multi-graph that corresponds to the C_{s_2} in our packing. Given the propositions in Section 3, we may assume that there exists at least two edges between V_{C_b} and $V_{C_{s_1}}$, as well as between V_{C_b} and $V_{C_{s_2}}$ because every vertex must have degree three. If we did not have at least two edges between V_{C_b} and $V_{C_{s_i}}$, for some i=1,2,

[1] The results of Brandt, Dickinson, Ellsworth, Kenkel, and Smith are technically for equal circle packings, but the arguments presented there also apply to circle packings that are not neccessarily equal.

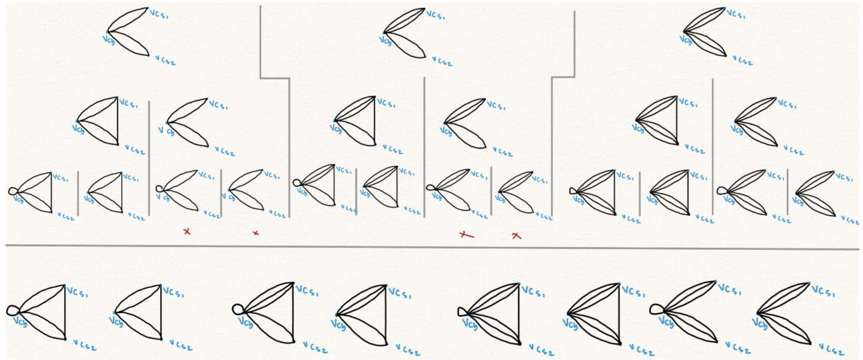


Figure 13: Limiting the possible combinatorial multi-graphs.

we would not have a degree three vertex at that C_{s_i} since there only exists at most one edge between $V_{C_{s_1}}$ and $V_{C_{s_2}}$. We also have from Proposition 3.9 that there exists a maximum of three edges between V_{C_b} and $V_{C_{s_1}}$, as well as between V_{C_b} and V_{C_b}. Thus, without loss of generality, this gives us three choices (4,5, and 6) for the number of edges incident to V_{C_b}. Refer to the top row of Figure 13. With each of these three choices, we can also choose to include the edge between $V_{C_{s_1}}$ and $V_{C_{s_2}}$, since from Proposition 3.2 the maximum number of tangencies between $V_{C_{s_1}}$ and $V_{C_{s_2}}$ on any flat torus is one. This doubles our number of choices from three to six. Again, refer to second to top row of Figure 13. Finally, by Proposition 3.4, we know that C_b can have at most one self tangency, so we can either have a loop on V_{C_b} or not; this doubles our number of choices from six to twelve. Refer to the middle top row of Figure 13. However, four of these choices can be eliminated either because not every vertex has degree three, or because the minimum number of edges is less than five. Thus we are left with 8 possible combinatorial multi-graphs. Refer to the bottom row of Figure 13 or Figure 12.

□

5 Embeddings of the Combinatorial Multi-Graphs on a Topological Torus

Figures 14, 15, 16 show all the possible embeddings of the combinatorial multi-graphs in Figure 12 onto a topological torus. These embeddings were found using a computer program, in which we inputted the combinatorial graphs in Figure 12 and the outputs were different ways in which the inputted combinatorial graphs embedded onto a topological torus. Now we begin to eliminate the embeddings which either do not correspond to packings of our three cricles on a flat torus T with $\frac{r_s}{r_b} = \sqrt{2} - 1$ or do not correspond to locally maximally dense packing with no free circles of our three circles on a flat torus T with $\frac{r_s}{r_b} = \sqrt{2} - 1$. The embeddings V3E06L01N01T11, V3E06L00N01T21, and V3E06L01N01T21 do not correspond to a packing of C_b, C_{s_1}, C_{s_2} on a flat torus T with $\frac{r_s}{r_b} = \sqrt{2} - 1$. The embeddings V3E05L01N01T12, V3E05L01N01T13, V3E06L00N01T11, V3E06L01N01T12 and V3E06L01N01T13, do not correspond to a locally maximally dense packing with no free circles of C_b, C_{s_1}, C_{s_2} on

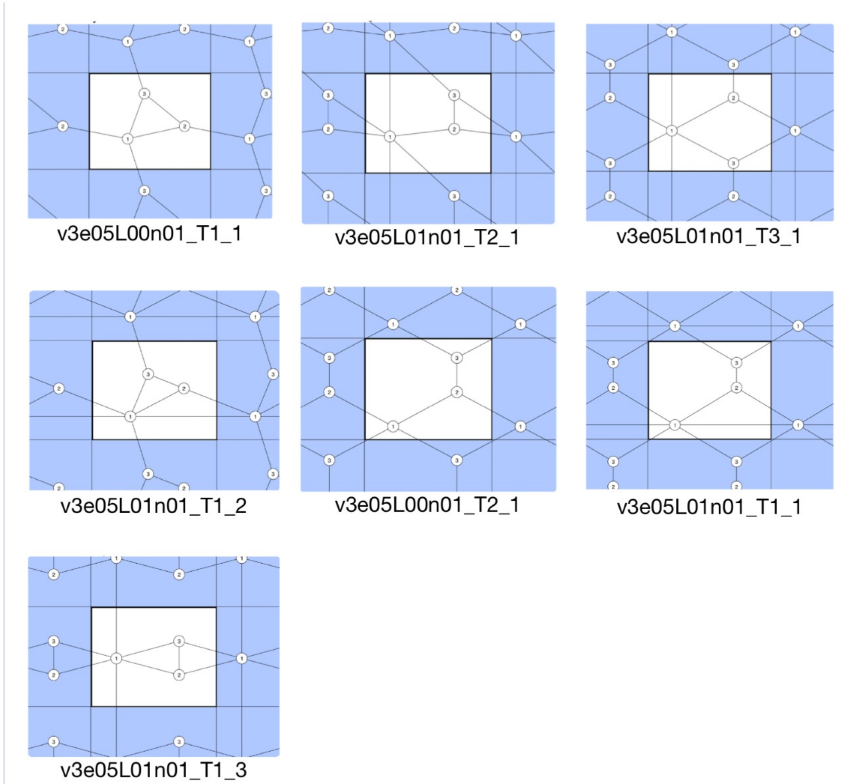


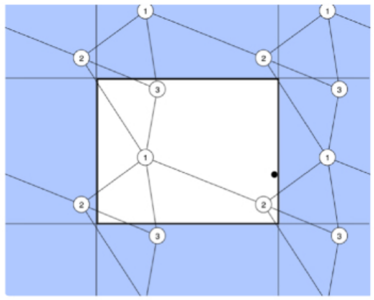
Figure 14: These are all the embeddings that correspond to the combinatorial multi-graphs in Figure 12 that have 5 edges in total, not counting loops.

a flat torus T with $\frac{r_s}{r_b} = \sqrt{2} - 1$. See Figure 17 for the embeddings which do not correspond to packings or do not correspond to locally maximally dense packing with no free circles of our three circles on a flat torus T with $\frac{r_s}{r_b} = \sqrt{2} - 1$. The remaining nine embeddings are analyzed on Mathematica. Now we prove that certain embeddings do not correspond to packings or do not correspond to locally maximally dense packing with no free circles of our three circles on a flat torus T with $\frac{r_s}{r_b} = \sqrt{2} - 1$.

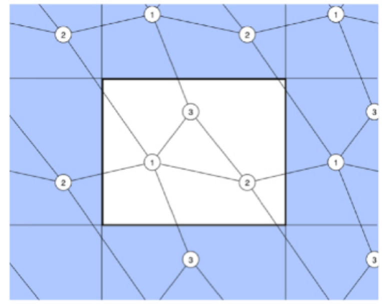
Proposition 5.1. *The embedding* V3E06L01N01T11*, pictured in Figure 15 and 18, does not correspond to a packing of circles C_b, C_{s_1}, C_{s_2} where $\frac{r_s}{r_b} = \sqrt{2} - 1$ on any flat torus T.*

Proof. We prove this by contradiction. Suppose that the embedding V3E06L00N01T11, pictured in Figure 15 and 18, did correspond to a packing of circles C_b, C_{s_1}, C_{s_2} where $\frac{r_s}{r_b} = \sqrt{2} - 1$ on some flat torus T. We can see that vertex 1, V_1, in the embedding corresponds to C_b on any flat torus T as Propositions 3.3 and 3.4 show that only C_b can have a self tangency on a flat torus T. Without loss of generality, let vertex 2, V_2, correspond to C_{s_2} and vertex 3, V_3, correspond to C_{s_1}. We lift the packing on T to the Euclidean plane.

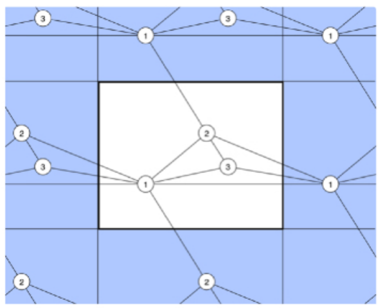
Refer to Figure 18 for the following argument. We have that $V_1V_3V_1$, the green triangle, is congruent to the $V_1V_2V_1$, the orange and green triangle, with the base of the orange triangle exactly where the base of the green triangle lies. Using the trivial translations of the plane, we may assume, without loss of generality, that the center of C_b is located at $(0, 0)$. The lifts of C_b are located at the lattice points of Λ. The base of our triangles then lie



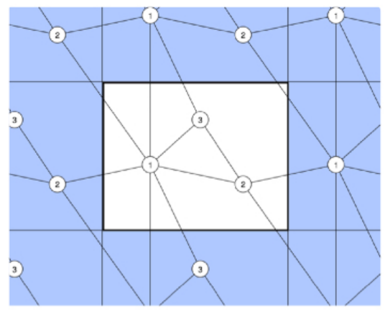
v3e06L00n01_T1_1



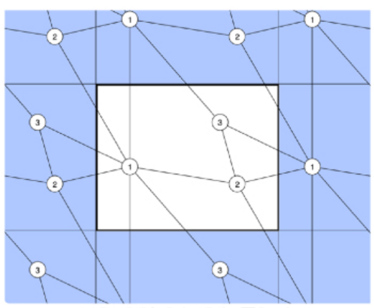
v3e06L00n01_T2_1



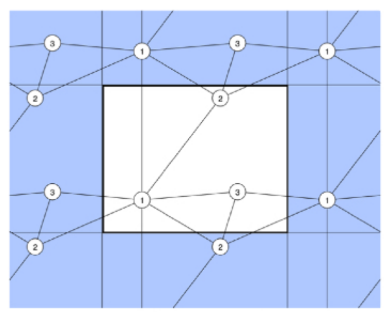
v3e06L01n01_T1_1



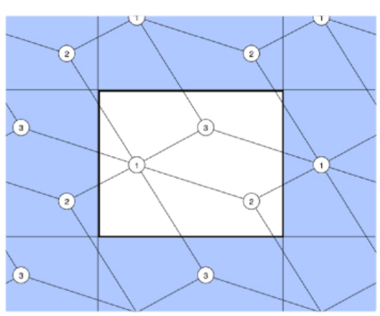
v3e06L01n01_T1_2



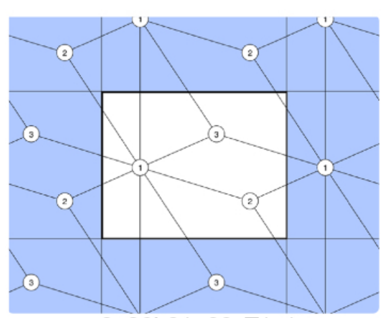
v3e06L01n01_T2_1



v3e06L01n01_T1_3



v3e06L00n02_T1_1



v3e06L01n02_T1_1

Figure 15: These are all the embeddings that correspond to the combinatorial multi-graphs in Figure 12 that have 6 edges in total, not counting loops.

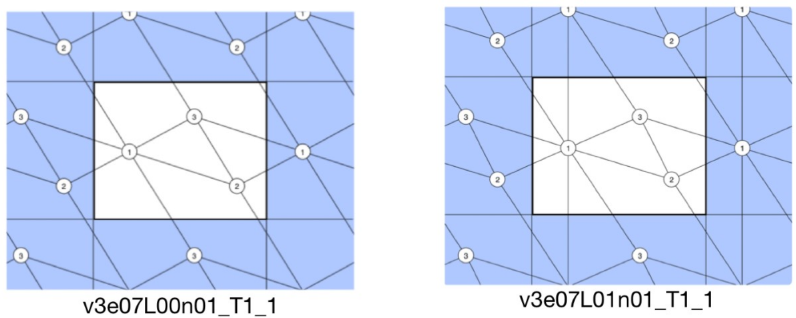


Figure 16: These are all the embeddings that correspond to the combinatorial multi-graphs in Figure 12 that have 7 edges in total, not counting loops.

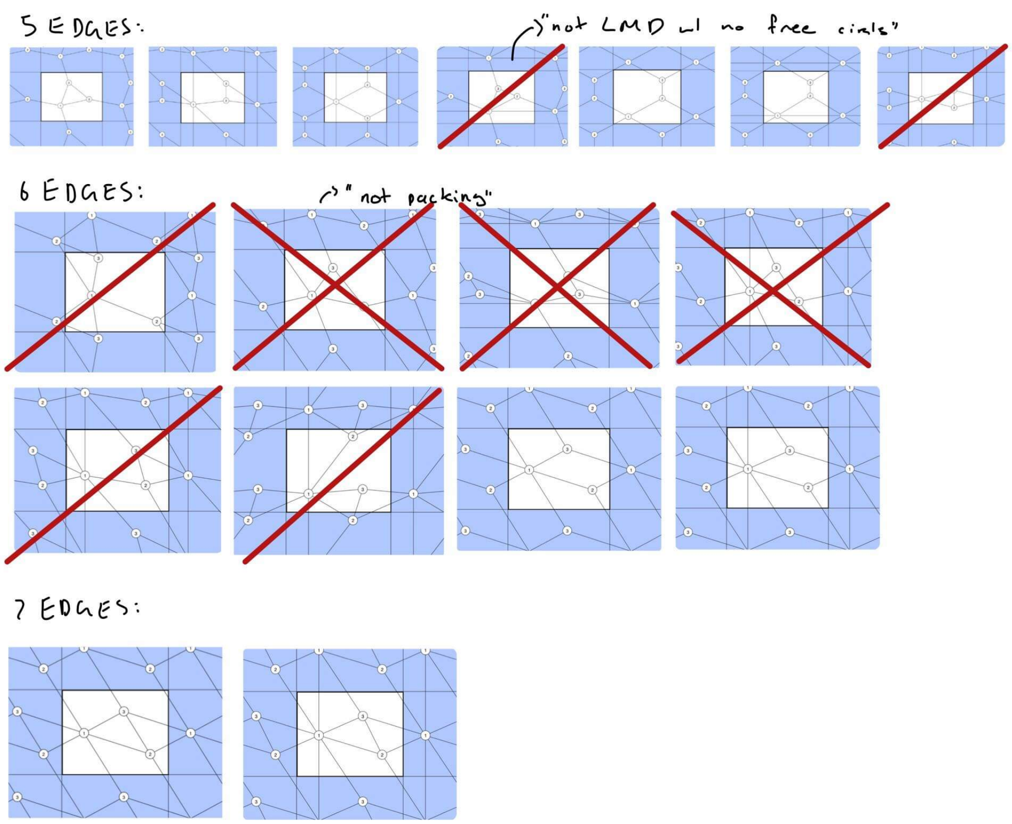


Figure 17: The embedding that is doubly crossed out does not correspond to a packing of our three circles on any flat torus T, and the embeddings which are singly crossed out do not correspond to a locally maximally dense packing with no free circles of our three circles on any flat torus T.

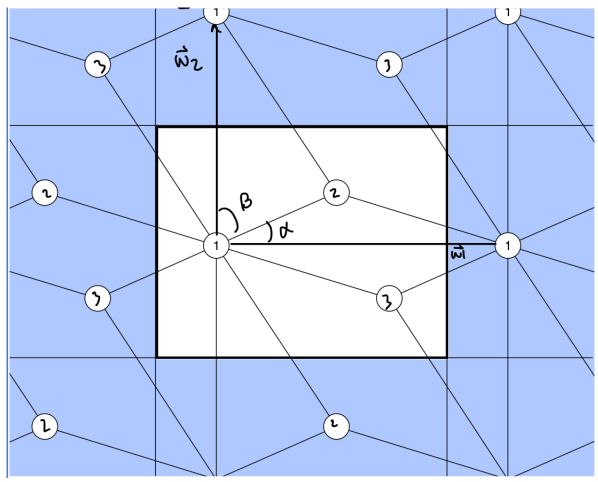


Figure 18: Refer to this figure for the argument in Proposition 5.1 which shows that V3E06L01N01T11 does not correspond to a packing of circles C_b, C_{s_1}, C_{s_2} where $\frac{r_s}{r_b} = \sqrt{2} - 1$ on any flat torus T.

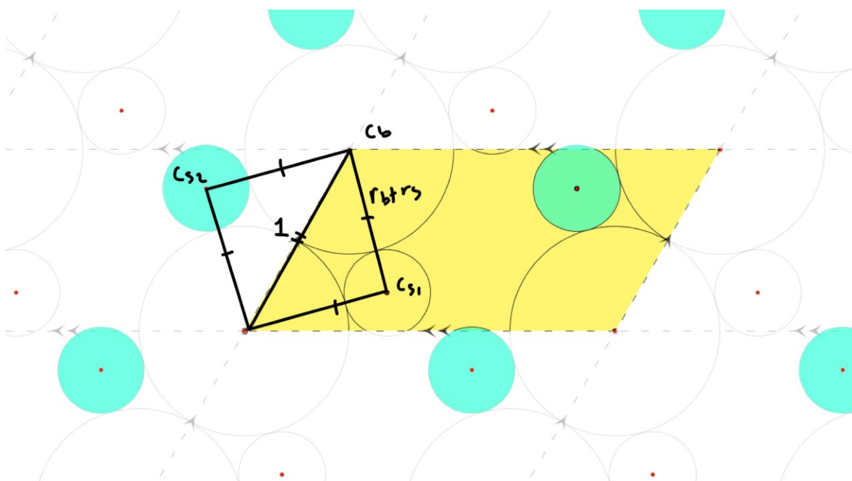


Figure 19: This packing does not correspond to V3E06L01N01T11.

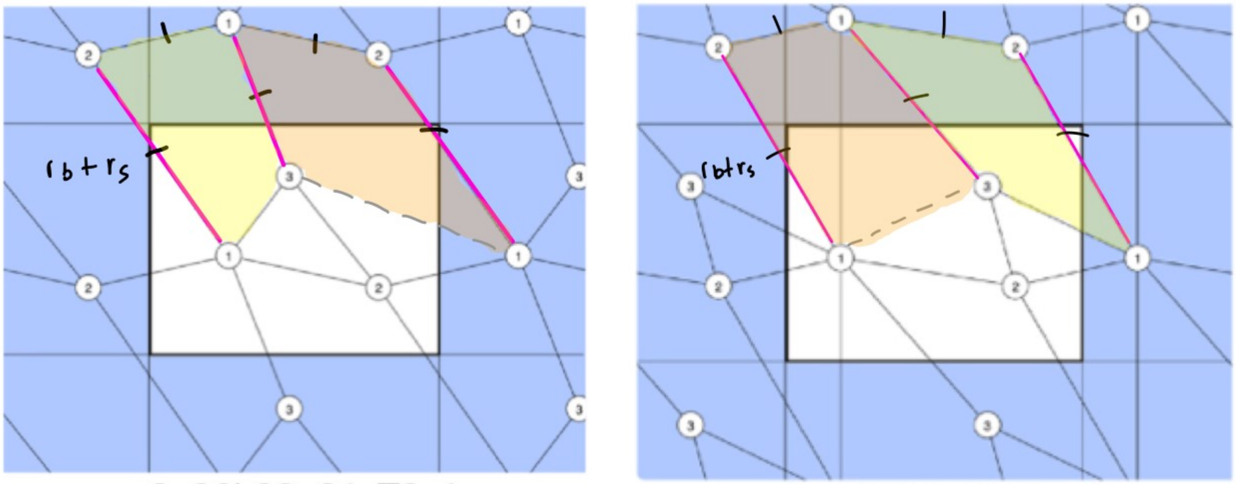
along $\vec{w_1}$, as by Proposition 3.5, because if C_b is self-tangent, it can be shown to be self-tangent only along $\vec{w_1}$ in the fundamental domain of Λ.

There exits centers of lifts of C_{s_1} and C_{s_1} in the fundamental domain of Λ. According to the geometry of congruent triangles that share a base, the centers may be reflected across $\vec{w_1}$ or lie on top of each other. However, the edge from V_1 to itself prevents this. See Figure 19.

Therefore, the centers must lie on top of each other. However, this implies that C_{s_1} and C_{s_2} would overlap, which is not allowed in a packing, and we have a contradiction. Thus, the embedding V3E06L01N01T12 does not correspond to a packing of C_b, C_{s_1}, C_{s_2} where $\frac{r_s}{r_b} = \sqrt{2} - 1$ on any flat torus T. □

Proposition 5.2. *The embeddings* V3E06L00N01T21 *and* V3E06L01N01T21*, as pictured in Figure 20 both do not correspond to packings of* C_b, C_{s_1}, C_{s_2}, *where* $\frac{r_s}{r_b} = \sqrt{2} - 1$, *on any flat torus* T.

Proof. We prove this by contradiction. First we show that V3E06L00N01T21 does not correspond to a packing of



v3e06L00n01_T2_1 v3e06L01n01_T2_1

Figure 20: Refer to this Figure for the argument in Proposition 5.2.

C_b, C_{s_1}, C_{s_2}, where $\frac{r_s}{r_b} = \sqrt{2}-1$, on any flat torus T. Then we use the fact that V3E06L00N01T21 is identical to V3E06L01N01T21 with the exception of the edges between vertex 1 and itself. Then, because our proof here does not rely on the edges between vertex 1 and itself, our same proof applies to showing both V3E06L00N01T21 and V3E06L01N01T21 do not correpsond to packings of C_b, C_{s_1}, C_{s_2}, where $\frac{r_s}{r_b} = \sqrt{2}-1$, on any flat torus T.

We have that vertex 1, V_1, corresponds to C_b in our packing as by Proposition 3.9 and 3.2, show that only C_b can be 2-tangent to two other circles in our packing. Without loss of generality, let vertex 2, V_2, correspond to C_{s_2} and vertex 3, V_3, correspond to C_{s_1}. We lift the packing on T to the Euclidean plane.

As shown in Figure 20, we have that $V_1V_2V_1V_3$, in yellow, is a rhombus as it a quadrilateral with all sides of equal length, $r_b + r_s$. We also have that the three pink line segments $\overline{V_1V_2}$, $\overline{V_3V_1}$, and $\overline{V_1V_2}$, are parallel. This is because $\overline{V_1V_2}$ and $\overline{V_3V_1}$ are opposite sides of a parallelogram and the edges form parallel lines with their respective lifts with when lifted to the plane, and so $\overline{V_1V_2}$ and $\overline{V_1V_2}$ are parallel. Then because being parallel is a transitive relation, we have that the three pink line segments $\overline{V_1V_2}$, $\overline{V_3V_1}$, and $\overline{V_1V_2}$, are parallel. This implies that the dashed distance between V_3 and V_1, as shown in Figure 20 must be $r_b + r_s$ as we have two congruent rhombii, $V_1V_2V_1V_3$, in yellow, and $V_3V_1V_2V_1$, in orange. However, this would imply that C_{s_1} is 3-tangent to C_b. which contradicts the embedding graph. Thus we have that V3E06L00N01T21 does not correspond to a packing of C_b, C_{s_1}, C_{s_2}, where $\frac{r_s}{r_b} = \sqrt{2}-1$, on any flat torus T.

We extend this argument to the embedding V3E06L01N01T21 to get that the embedding V3E06L01N01T21 also does not correspond to a packing of C_b, C_{s_1}, C_{s_2}, where $\frac{r_s}{r_b} = \sqrt{2}-1$, on any flat torus T. □

Now we will show in the following propositions that the embeddings V3E05L01N01T12, V3E05L01N01T13, V3E06L00N01T11, V3E06L01N01T12 and V3E06L01N01T13 do not correspond to locally maximally dense packing with no free circles of C_b, C_{s_1}, C_{s_2} on a flat torus T with $\frac{r_s}{r_b} = \sqrt{2}-1$. We will do this using a result from [3] which

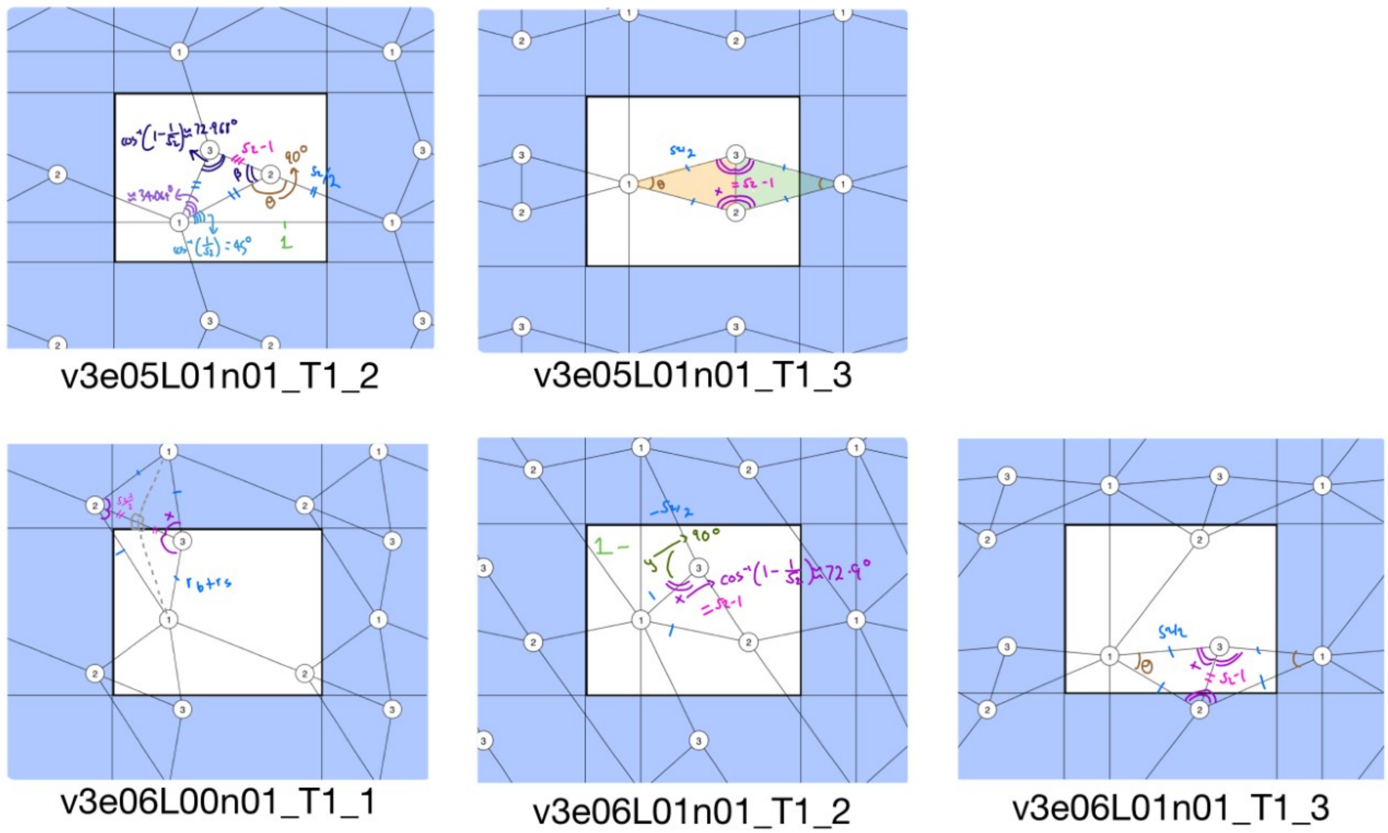


Figure 21: Refer to this figure for the notation used in Proposition 5.3.

tells us that if a circle in a packing has its points of tangency contained in a closed semi-circle, then that packing is not a locally maximally dense packing of circles on a flat torus with no free circles.

Proposition 5.3. *The embeddings* V3E05L01N01T12, V3E05L01N01T13, V3E06L00N01T11, V3E06L01N01T12, *and* V3E06L01N01T13*, pictured in Figure 21, do not correspond to locally maximally dense packings with no free circles of circles C_b, C_{s_1}, C_{s_2} where $\frac{r_s}{r_b} = \sqrt{2} - 1$, on any flat torus T.*

Proof. We prove this by contradiction. Suppose that the embeddings V3E05L01N01T12, V3E05L01N01T13, V3E06L00N01T11, V3E06L01N01T12 and V3E06L01N01T13, pictured in Figure 21, did correspond to locally maximally dense packings with no free circles, of circles C_b, C_{s_1}, C_{s_2}, where $\frac{r_s}{r_b} = \sqrt{2} - 1$, on some flat torus T. We can see that in each of these embeddings, vertex 1, which we denote by V_1 from here onwards, corresponds to C_b on any flat torus T by either Propositions 3.4 and 3.3, which show that only C_b can have a self tangency on any flat torus T or by Propositions 3.4 and 3.3, which show that only C_b can be 2-tangent to both other circles in our packing. Without loss of generality, let V_2 correspond to C_{s_2} and V_3 correspond to C_{s_1}. We lift the packing on T to the Euclidean plane. We will show that each of the embeddings has at least one circle whose points of tangency are contained is a closed semi-cricle. Refer to 21 for the following notation.

We start with V3E05L01N01T12. Because C_b is self-tangent, by Proposition 3.5, we have that $r_b = \frac{1}{2}$, and thus $r_s = \frac{1}{2}(\sqrt{2} - 1)$. We have that in our packing, there exists two isosceles triangles as shown in Figure 21. We get that the pink angle α is $\frac{\pi}{2}$ and that the brown angle β is $\arccos(1-\frac{1}{\sqrt{2}}) \approx 72.968°$. That is, $\alpha+\beta<\pi$. Thus, C_{s_1} has its points of tangency contained in a closed semi circle.

Now we move to V3E05L01N01T13. By Proposition 3.5, because C_b is self-tangent, we have that $r_b=\frac{1}{2}$ and so $r_s=\frac{1}{2}(\sqrt{2}-1)$. We have that in our packing, there exists two congruent isosceles triangles as shown in the embedding. We solve for x and get that $2x=\pi-2\arcsin(1-\frac{1}{\sqrt{2}})$. That is, $2x<\pi$. This tells us that C_{s_1} has its points of tangency contained in a closed semi-circle.

Next we consider V3E06L00N01T11. We have that in our packing, there exits two congruent triangles as shown in Figure 21. We solve for angle x and get that $x=\arccos(1-\frac{1}{\sqrt{2}}) \approx 72.968°$; thus $2x<\pi$. Thus, C_{s_1} has its points of tangency contained in a closed semi circle.

Next we consider V3E06L01N01T12. Because C_b is self-tangent, by Proposition 3.5, we have that $r_b=\frac{1}{2}$, and thus $r_s=\frac{1}{2}(\sqrt{2}-1)$. We have that in our packing, there exists two isosceles triangles as shown in Figure 21. We solve for angles y and x and get that $y=\frac{\pi}{2}$ and $x=\arccos(1-\frac{1}{\sqrt{2}}) \approx 72.968°$; that is $x+y<\pi$. Thus, C_{s_1} has its points of tangency contained in a closed semi circle.

Finally we consider V3E06L01N01T13. Because C_b is self-tangent, by Proposition 3.5, we have that $r_b=\frac{1}{2}$, and thus $r_s=\frac{1}{2}(\sqrt{2}-1)$. We have that in our packing, there exists two congruent isosceles triangles as shown in Figure 21. We solve for angle x and get that $2x=\pi-2\arcsin(1-\frac{1}{\sqrt{2}})$. Thus, C_{s_1} has its points of tangency contained in a closed semi circle.

Now we use Proposition 3.3 from the paper by Brandt, Dickinson, Ellsworth, Kenkel, and Smith [3], which tells us that if a packing contains a circle whose points of tangency are contained is a closed semi-cricle, then that packing is not locally maximally dense on a flat torus with no free circles. Therefore we have a contradiction to our assumption.

□

In the next section we will analyze the remaining nine embeddings as shown in Figure 22.

6 Analyzing Embeddings on Flat Tori

In this section, we will separately analyze the remaining embeddings shown in Figure 22. Full details can be found in the Mathematica documents. We will give an overview of our steps and results in this section. Our method on Mathematica consists of first finding the centers of the circles in our embedding graphs in terms of some parameters, which will vary case by case. We also find the basis vectors $\vec{w_1}$ and $\vec{w_2}$ in terms of these parameters. We then verify that these centers and lattice vectors in fact do give us the right embedding graphs by checking edge lengths. Then we impose bounds on the parameters so that the bounded parameters with those centers give us the correct embedding graphs, locally maximally dense packings with no free circles, and do not over count any packings. We then verify that the packings given by those bounded parameters and centers are in fact locally maximally dense

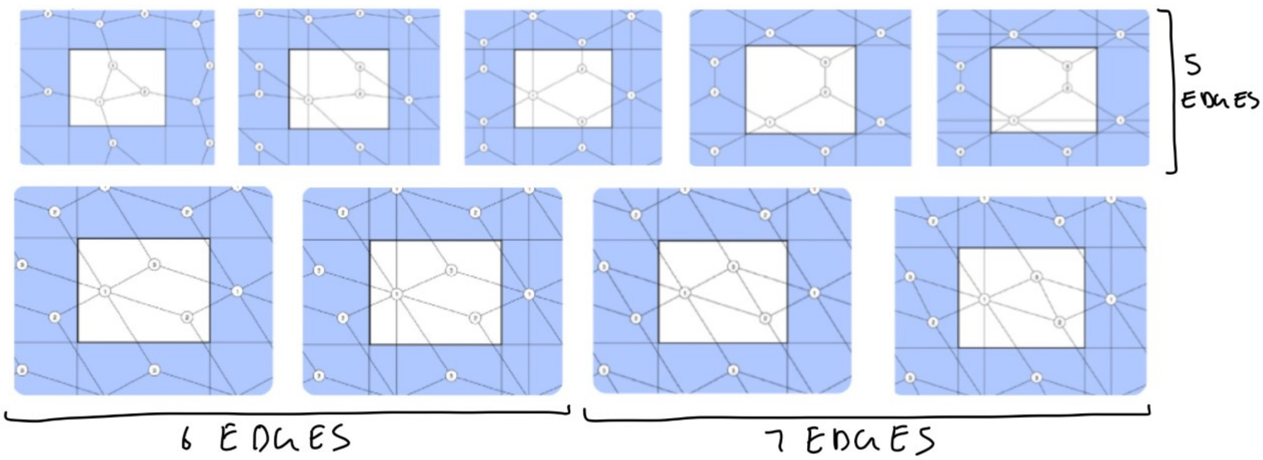


Figure 22: Out of the 17 embeddings, these are the remaining 9 embeddings that were not eliminated in Section 5.

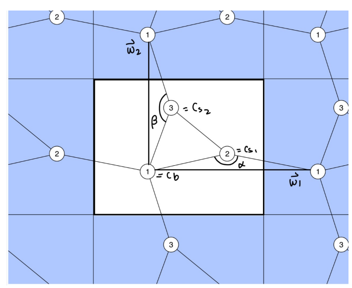


Figure 23: We choose α and β to be the angles between the segments $\overline{V_1V_2}$ and $\overline{V_2V_1}$, and $\overline{V_1V_3}$ and $\overline{V_3V_1}$, respectively. We note that V_1 must correspond to C_b and we let, without loss of generality, V_2 correspond to C_{s_1} and V_3 correspond to C_{s_2}.

packings with no free circles. Then we associate those packings to the region in the moduli space that gives us those packings. Finally we calculate and compare the denisty of those packings.

We have three embeddings (V3E05L00N01T11, V3E05L00N01T21, V3E06L00N02T11) that will have locally maximally dense packings with no free circles that correspond to two dimensional open regions in the moduli space. The other six embeddings will have locally maximally dense packings with no free circles that correspond to one dimensional closed regions in the moduli space, bordering the open two dimensional open regions.

6.1 V3E05L00N01T11

We start by analyzing the first embedding which has locally maximally dense packings with no free cricles that occupy an open two dimensional region in the moduli space: V3E05L00N01T11, as shown in Figure 23. We first choose α and β to be the angles between the segments $\overline{V_1V_2}$ and $\overline{V_2V_1}$, and $\overline{V_1V_3}$ and $\overline{V_3V_1}$, respectively, as pictured in Figure 23. Our choice of $\vec{w_1}$ and $\vec{w_2}$ is arbitrary. We note that V_1 must correspond to C_b as Propositions 3.2 and 3.9 show us that only C_b can be 2-tangent to two other circles in our packing. Then, without loss of generality, we let V_2 correspond to C_{s_1} and V_3 correspond to C_{s_2}.

We lift the packing on T to the Euclidean plane. Using trivial translation of the plane we translate the center

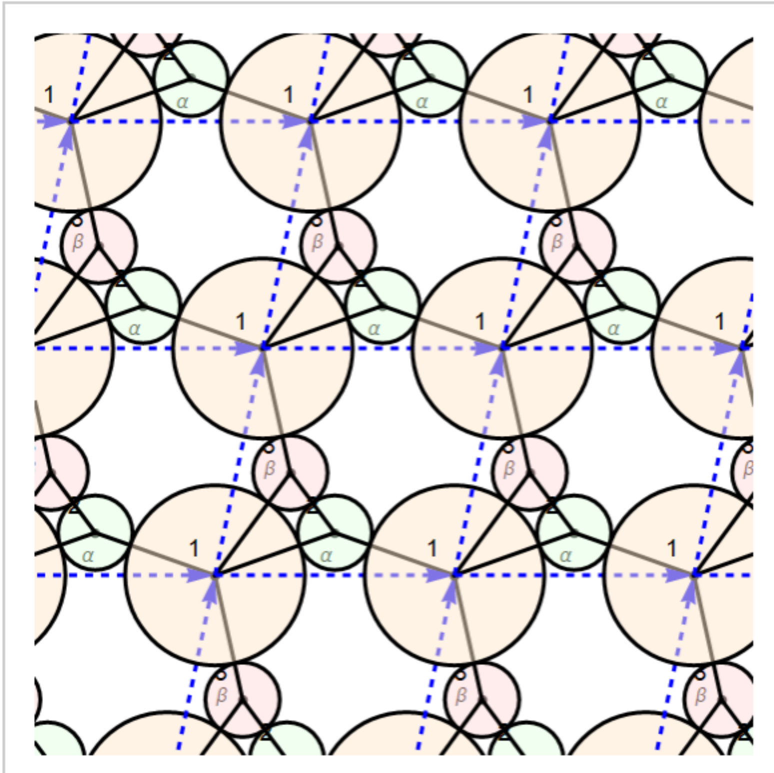


Figure 24: This figure shows what a typical locally maximally dense packing that corresponds to V3E05L00N01T11 looks like.

of C_b to $(0, 0)$. We then find the centers of the other circles and the endpoints of $\vec{w_1}$ and $\vec{w_2}$. We scale the whole packing so that $r_b = \frac{1}{2}$.

We verify that these centers and endpoints of the lattice vectors in fact do give us the right embeddings on Mathematica by checking the distances between the centers of the lifts, i.e., the edge lengths in our embeddings. Since we chose to scale r_b to $\frac{1}{2}$, all the edge lengths should be of magnitude $r_b + r_s = \frac{1}{\sqrt{2}}$, or $r_s + r_s = \sqrt{2} - 1$, or $r_b + r_b = 1$, where the magnitude depends on the tangencies.

Now we impose bounds on α and β so that the bounded parameters with the centers and endpoints we found will give us the correct embedding graphs, locally maximally dense packings with no free circles, and will not over count any packings. Figure 24 shows what a general locally maximally dense packing that corresponds to V3E05L00N01T11 looks like. We verify that these packings are in fact locally maximally dense with no freee circles by checking if the packings are properly stressed. The stresses are in the Mathematica document.

Now we determine what region of the moduli space the locally maximally dense packings with no free circles that correspond to V3E05L00N01T11 occupy. We get the region depicted in Figure 25. See Mathematica document for the stresses that verify this.

In the next corollary we approximate the locally maximally dense packing with no free circles that has the least density that corresponds to the embedding graph of V3E05L00N01T11.

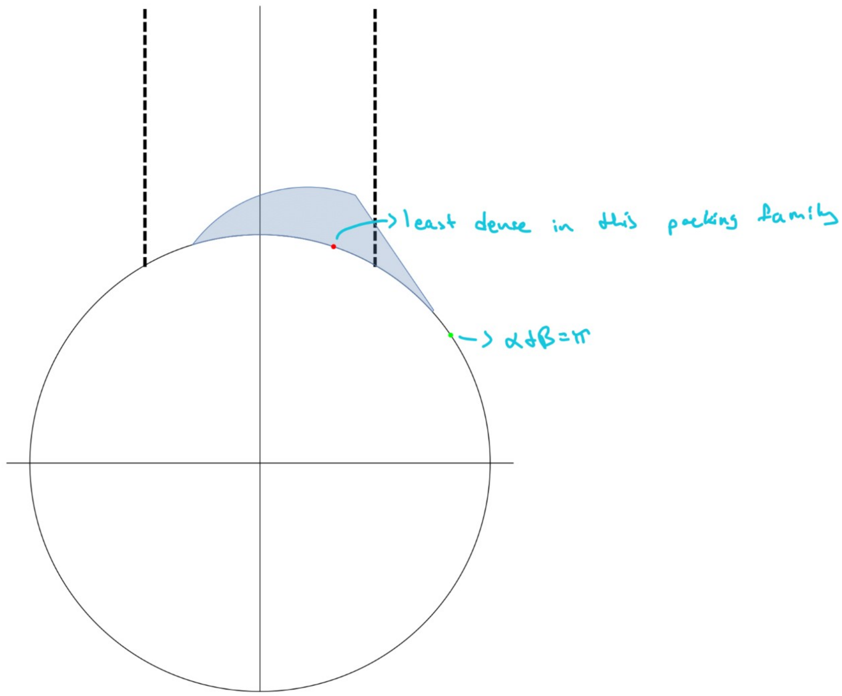


Figure 25: This is the region of the moduli space that the locally maximally dense packings with no free circles that correspond to the embedding V3E05L00N01T11 occupy.

Corollary. *For the locally maximally dense packing with no free circles that has the least density that corresponds to the embedding graph of V3E05L00N01T11,* $\alpha \approx 2.49$ *and* $\beta \approx 2.49$*. This packing occupies the region of the moduli space denoted by the red dot as shown in Figure 25.*

We note that the region in Figure 25 is not contained in the moduli space strip. We can amend this using reflections and inversion, both of which preserve density. We get the regions depicted in Figure 26.

We observe that the locally maximally dense packings with no free circles that correspond to the embedding occupy an open two dimensional region in the moduli space. The locally maximally dense packings with no free circles that correspond to the embedding V3E05L01N01T21 border a part of this two dimensional region. This is because if we add the edges as shown in Figure 27 to V3E05L00N01T11 we get V3E05L01N01T21. We will further analyze V3E05L01N01T21 in the following subsection.

6.2 V3E05L01N01T21

We now analyze the embedding V3E05L01N01T21, as shown in Figure 23. We first choose α and β to be the angles between the line segments $\overline{V_1V_2}$ and $\overline{V_2V_1}$, and $\overline{V_1V_3}$ and $\overline{V_3V_1}$ respectively, as pictured in Figure 28. We keep the same choice of $\vec{w_1}$ and $\vec{w_2}$ as we had for V3E05L00N01T11. We note that V_1 must correspond to C_b as Propositions 3.4 and 3.3 show us that only C_b can self-tangent in our packing. Then, without loss of generality, we let V_2 correspond to C_{s_1} and V_3 correspond to C_{s_2}.

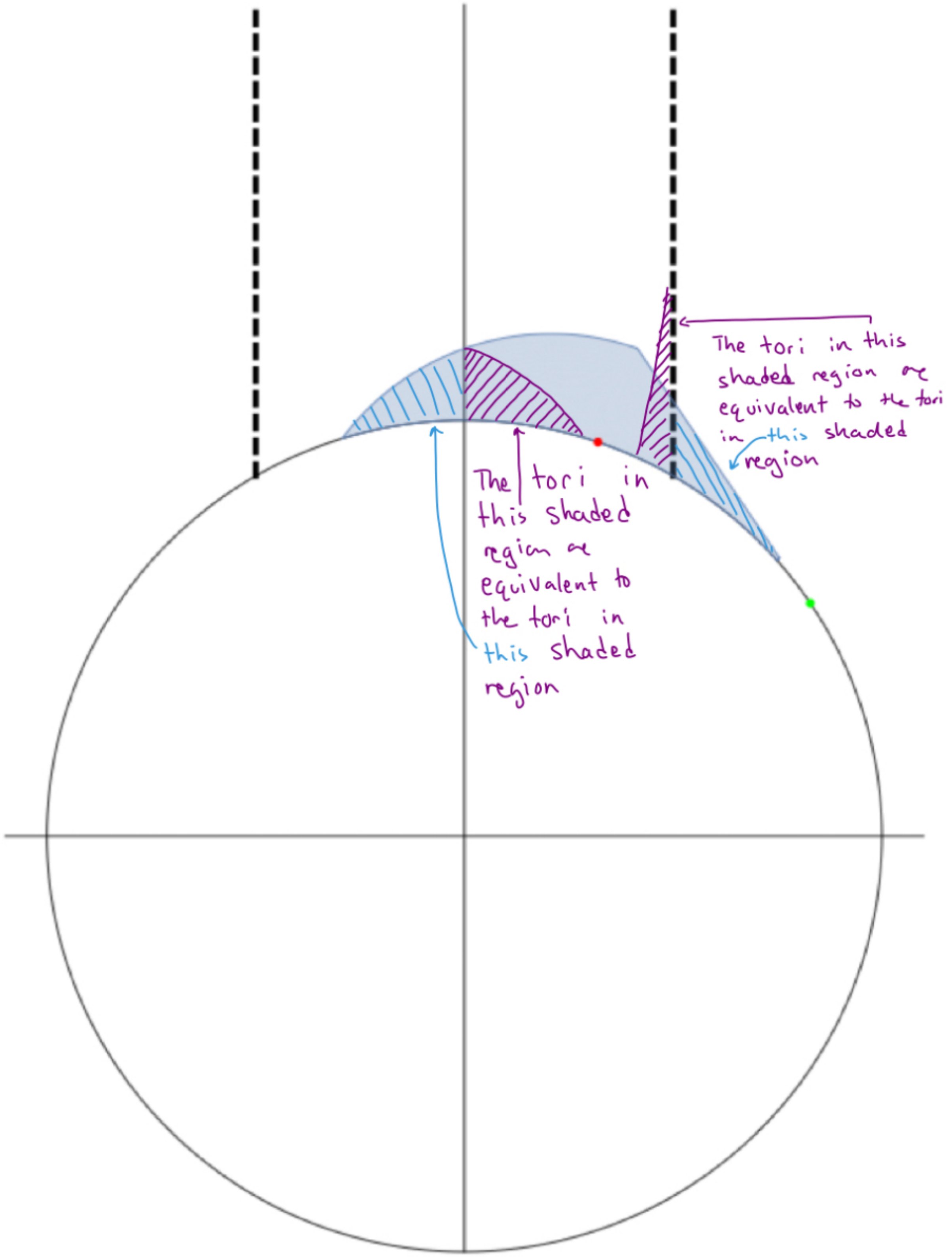


Figure 26: This is the region of the "moduli space" that the locally maximally dense packings with no free circles that correspond to the embedding V3E05L00N01T11 occupy. Will- I can't fix this picture because I don't have the corresponding Mathematica results. Once I get them from Dan, I will fix it and resend the document.

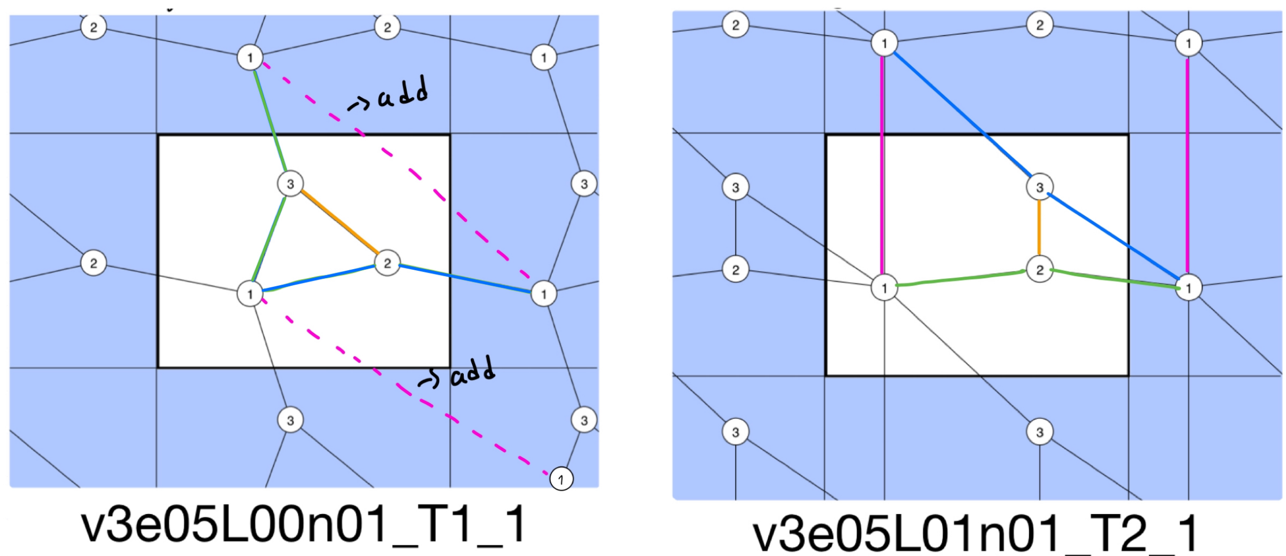


Figure 27: If we add the dotted pink edges to V3E05L00N01T11 we get V3E05L01N01T2.

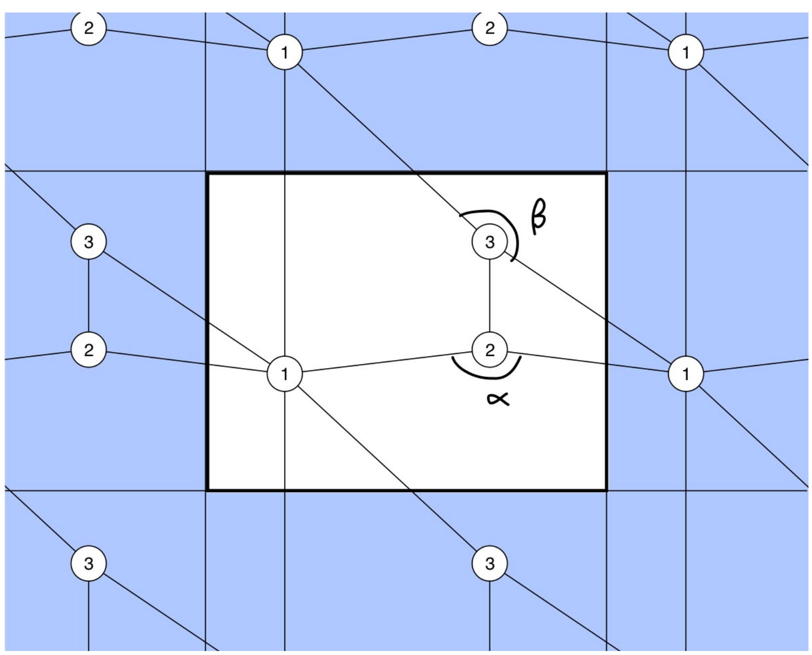


Figure 28: We choose α and β to be the angles between line segments $\overline{V_1V_2}$ and $\overline{V_2V_1}$, and $\overline{V_1V_3}$ and $\overline{V_3V_1}$ respectively. We note that V_1 must correspond to C_b and we let, without loss of generality, V_2 correspond to C_{s_1} and V_3 correspond to C_{s_2}.

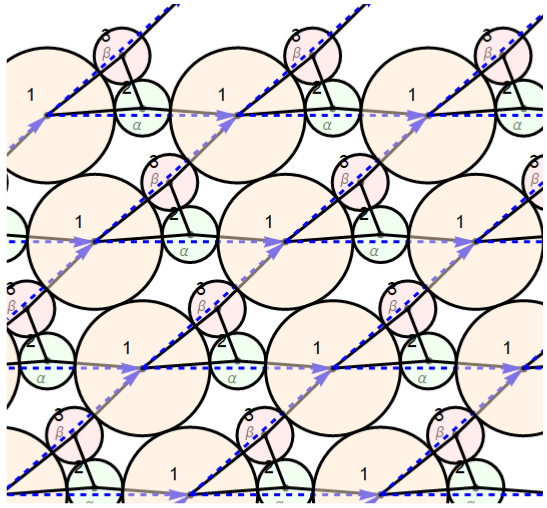


Figure 29: This figure shows what a typical locally maximally dense packing that corresponds to V3E05L01N01T21 looks like.

We lift the packing on T to the Euclidean plane. Using trivial translation of the plane we translate the center of C_b to $(0, 0)$. We then find the centers of the other circles and the endpoints of $\vec{w_1}$ and $\vec{w_2}$. We note that by Proposition 3.5, because C_b is self-tangent, r_b must be $\frac{1}{2}$.

We verify that these centers and endpoints of the lattice vectors in fact do give us the right embeddings on Mathematica by checking the distances between the centers of the lifts, i.e., the edge lengths in our embeddings. Since we have that $r_b = \frac{1}{2}$, all the edge lengths should be of magnitude $r_b + r_s = \frac{1}{\sqrt{2}}$, or $r_s + r_s = \sqrt{2} - 1$, or $r_b + r_b = 1$, where the magnitude depends on the tangencies.

Now we impose bounds on α and β so that the bounded parameters with the centers and endpoints we found will give us the correct embedding graphs, locally maximally dense packings with no free circles, and will not over count any packings. Figure 29 shows what a general locally maximally dense packing that corresponds to V3E05L01N01T21 looks like. We verify that these packings are in fact locally maximally dense with no freee circles by checking if the packings are properly stressed. See Mathematica document for the stresses.

Now we determine what region of the moduli space the locally maximally dense packings with no free circles that correspond to V3E05L01N01T21 occupy. We get the region depicted in Figure 30.

The region in Figure 30 is not contained in the moduli space strip. We can amend this using reflections and the negative of inversion, both of which preserve lattices. We get the region depicted in Figure 31.

We note that, as we said previously, the locally maximally dense packings with no free circles that correspond to the embedding V3E05L01N01T21 occupy a closed one dimensional region in the moduli space, bordering part of the open two dimensional region that the locally maximally dense packings with no free circles that correspond to the embedding V3E05L00N01T11.

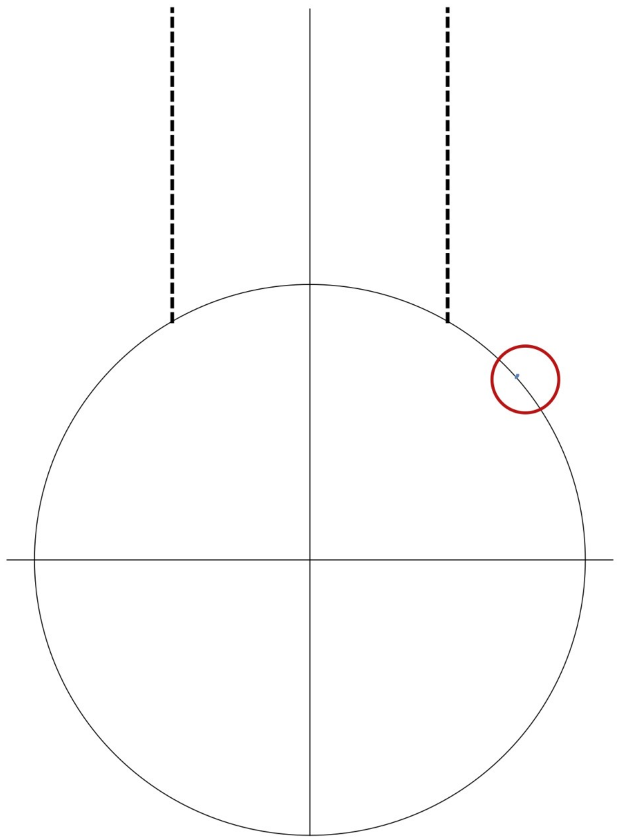


Figure 30: The tiny one dimensional circled blue region is the region of the moduli space that the locally maximally dense packings with no free circles that correspond to the embedding V3E05L01N01T21 occupy.

V3E05L01N01T21inms.jpg

Figure 31: This is an equivalent region in the moduli space strip, that the locally maximally dense packings with no free circles that correspond to the embedding V3E05L01N01T21 occupy. Will- I am missing this picture. I will add it once I get it from Dan.

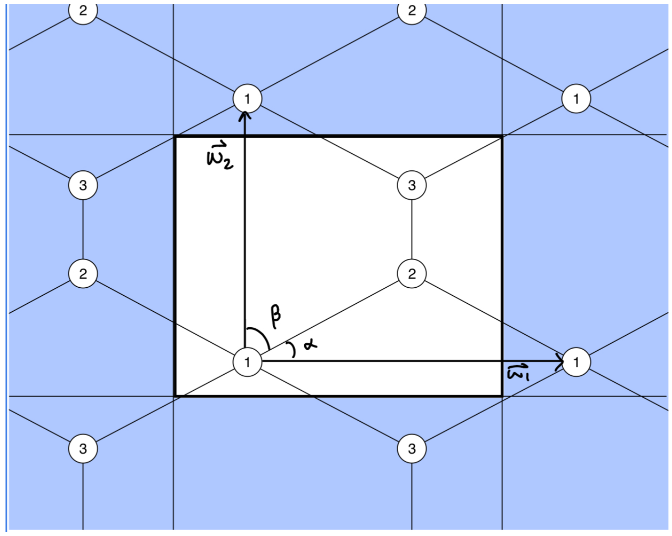


Figure 32: We choose α and β to be the angles between the segments $\overline{V_1V_2}$ and $\vec{w_1}$, and $\overline{V_1V_2}$ and $\vec{w_2}$, respectively. We note that V_1 must correspond to C_b and we let, without loss of generality, V_2 correspond to C_{s_1} and V_3 correspond to C_{s_2}.

6.3 V3E05L00N01T21

Now we analyze the second embedding which has locally maximally dense packings with no free cricles that occupy an open two dimensional region in the moduli space: V3E05L00N01T21, as shown in Figure 32. We first choose α and β to be the angles between the segments $\overline{V_1V_2}$ and $\vec{w_1}$, and $\overline{V_1V_2}$ and $\vec{w_2}$, respectively, as pictured in Figure 32. Our choice of $\vec{w_1}$ and $\vec{w_2}$ is arbitrary. We note that V_1 must correspond to C_b as Propositions 3.2 and 3.9 show us that only C_b can be 2-tangent to two other circles in our packing. Then, without loss of generality, we let V_2 correspond to C_{s_1} and V_3 correspond to C_{s_2}.

We lift the packing on T to the Euclidean plane. Using trivial translation of the plane we translate the center of C_b to $(0, 0)$. We then find the centers of the other circles and the endpoints of $\vec{w_1}$ and $\vec{w_2}$. We scale the whole packing so that $r_b = \frac{1}{2}$.

We verify that these centers and endpoints of the lattice vectors in fact do give us the right embeddings on Mathematica by checking the distances between the centers of the lifts, i.e., the edge lengths in our embeddings. Since we chose to scale r_b to $\frac{1}{2}$, all the edge lengths should be of magnitude $r_b + r_s = \frac{1}{\sqrt{2}}$, or $r_s + r_s = \sqrt{2} - 1$, or $r_b + r_b = 1$, where the magnitude depends on the tangencies.

Now we impose bounds on α and β so that the bounded parameters with the centers and endpoints we found will give us the correct embedding graphs, locally maximally dense packings with no free circles, and will not over count any packings. Figure 33 shows what a general locally maximally dense packing that corresponds to V3E05L00N01T21 looks like. We verify that these packings are in fact locally maximally dense with no freee circles by checking if the packings are properly stressed. See Mathematica document for the stresses.

Now we determine what region of the moduli space the locally maximally dense packings with no free circles

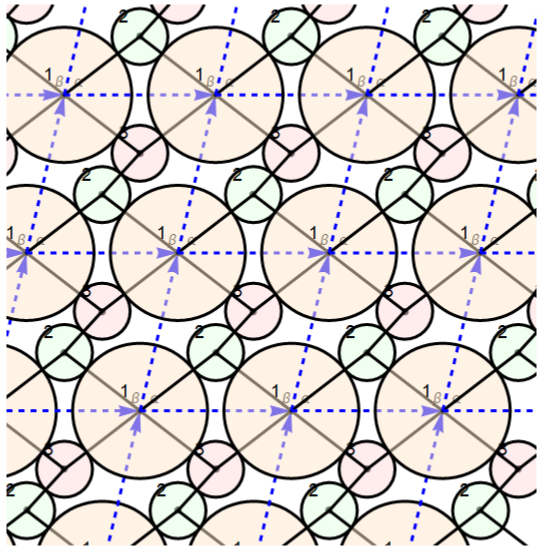


Figure 33: This figure shows what a typical locally maximally dense packing that corresponds to V3E05L00N01T21 looks like.

that correspond to V3E05L00N01T21 occupy. We get the region depicted in Figure 34.

Remark. *The overlapping in Figure 34 means that there is a region where there are tori with two locally maximally dense packings that are not homotopic.*

We note that the locally maximally dense packings with no free circles that correspond to the embedding occupy an open two dimensional region in the moduli space. The locally maximally dense packings with no free circles that correspond to the embeddings V3E05L01N01T11 and V3E05L01N01T31 border parts of this two dimensional region. This is because V3E05L01N01T11 and V3E05L01N01T31 are identical to V3E05L00N01T21 except for the fact that V3E05L01N01T11 and V3E05L01N01T31 have a self-tangency of V_1. We will further analyze V3E05L01N01T11 and V3E05L01N01T31 in the following subsections.

6.4 V3E05L01N01T11

We now analyze the embedding V3E05L01N01T11, as shown in Figure 35. We first choose β to be the angles between line segments $\overline{V_1V_2}$ and $\vec{w_1}$, as pictured in Figure 35. Our other parameter will be the length of $\vec{w_2}$ which we will denote by L. Our positions of $\vec{w_1}$ and $\vec{w_2}$ are also shown in Figure 35. We note that V_1 must correspond to C_b as Propositions 3.4 and 3.3 show us that only C_b can self-tangent in our packing. Then, without loss of generality, we let V_2 correspond to C_{s_1} and V_3 correspond to C_{s_2}.

We lift the packing on T to the Euclidean plane. Using trivial translation of the plane we translate the center of C_b to $(0, 0)$. We then find the centers of the other circles and the endpoints of $\vec{w_1}$ and $\vec{w_2}$. We note that by Proposition 3.5, because C_b is self-tangent, r_b must be $\frac{1}{2}$.

We verify that these centers and endpoints of the lattice vectors in fact do give us the right embeddings on

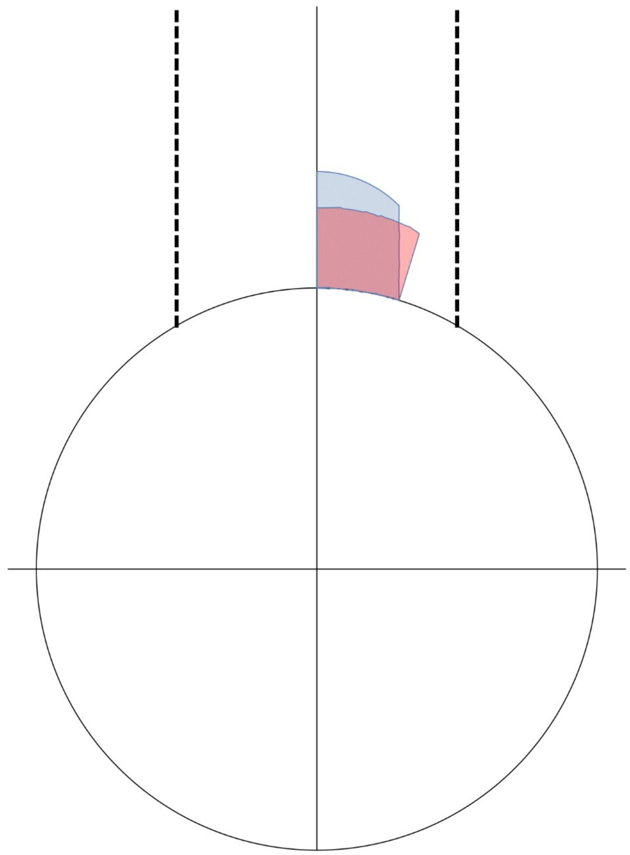


Figure 34: This is the region of the moduli space that the locally maximally dense packings with no free circles that correspond to the embedding V3E05L00N01T21 occupy.

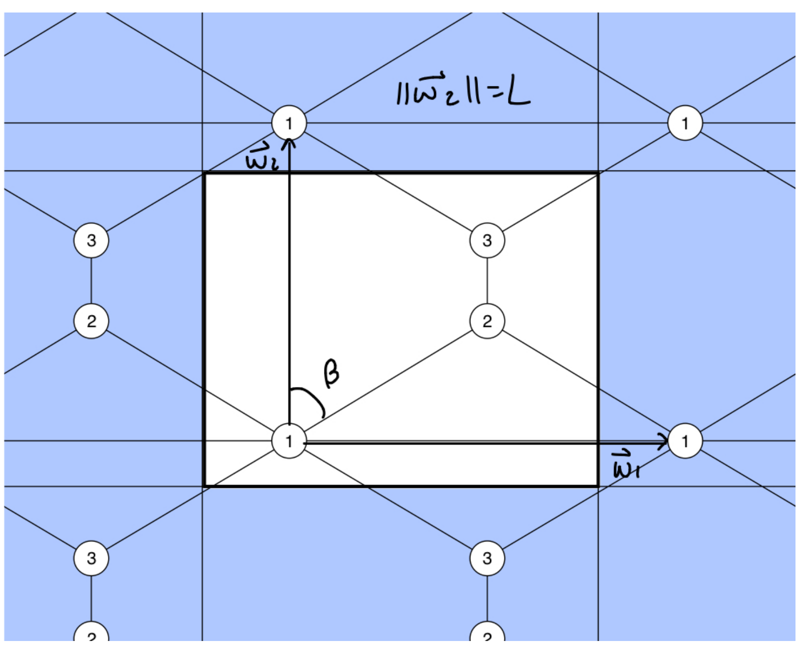


Figure 35: We choose β to be the angles between line segments $\overline{V_1V_2}$ and $\vec{w_1}$. We note that V_1 must correspond to C_b and we let, without loss of generality, V_2 correspond to C_{s_1} and V_3 correspond to C_{s_2}.

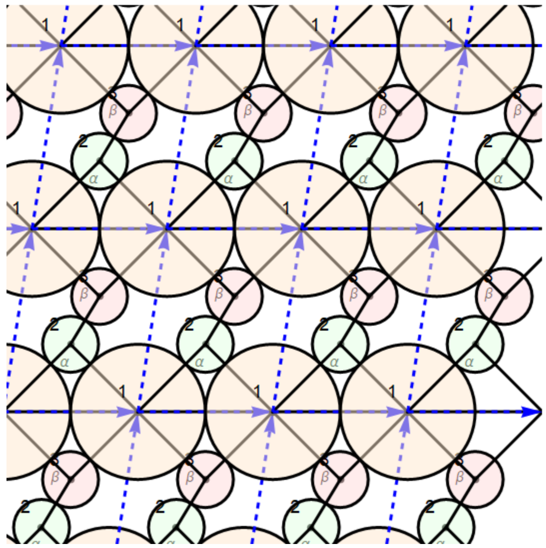


Figure 36: This figure shows what a typical locally maximally dense packing that corresponds to V3E05L01N01T11 looks like.

Mathematica by checking the distances between the centers of the lifts, i.e., the edge lengths in our embeddings. Since we have that $r_b = \frac{1}{2}$, all the edge lengths should be of magnitude $r_b + r_s = \frac{1}{\sqrt{2}}$, or $r_s + r_s = \sqrt{2} - 1$, or $r_b + r_b = 1$, where the magnitude depends on the tangencies.

Now we impose bounds on β and L so that the bounded parameters with the centers and endpoints we found will give us the correct embedding graphs, locally maximally dense packings with no free circles, and will not over count any packings. Figure 36 shows what a general locally maximally dense packing that corresponds to V3E05L01N01T11 looks like. We verify that these packings are in fact locally maximally dense with no freee circles by checking if the packings are properly stressed. See Mathematica document for the stresses.

Now we determine what region of the moduli space the locally maximally dense packings with no free circles that correspond to V3E05L01N01T11 occupy. We get the region depicted in Figure 37.

We note that, as we said previously, the locally maximally dense packings with no free circles that correspond to the embedding V3E05L01N01T11 occupy a closed one dimensional region in the moduli space, bordering part of the open two dimensional region that the locally maximally dense packings with no free circles that correspond to the embedding V3E05L00N01T21 occupy.

In this next subsection we analyze the other embedding whose locally maximally dense packings with no free circles occupy a closed one dimensional region that borders the open two dimensional region that the locally maximally dense packings with no free circles that correspond to the embedding V3E05L00N01T21 occupy.

6.5 V3E05L01N01T31

We now analyze the embedding V3E05L01N01T11, as shown in Figure 38. We first choose θ to be the angles between $\vec{w_1}$ and $\vec{w_2}$, as pictured in Figure 38. Our other parameter will be the length of $\vec{w_2}$ which we will denote

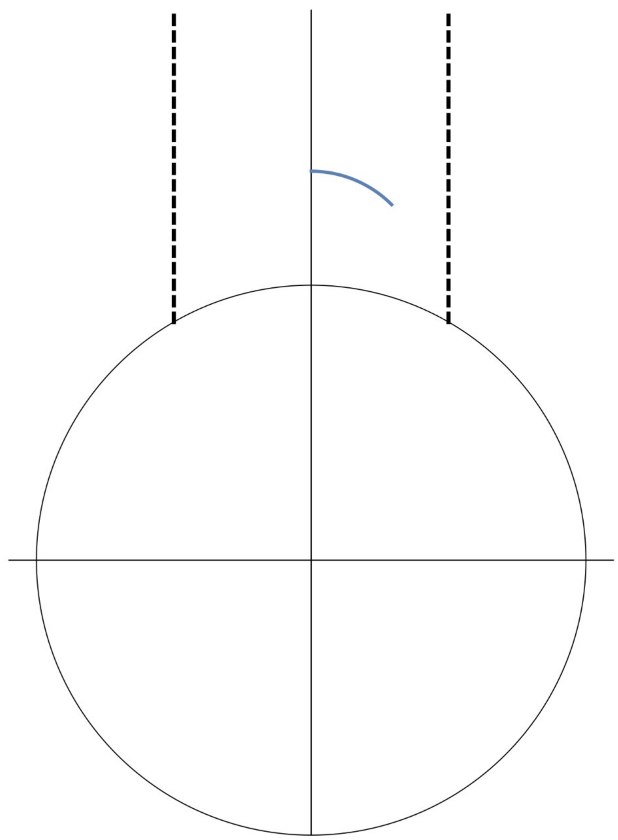


Figure 37: The one dimensional blue region is the region of the moduli space that the locally maximally dense packings with no free circles that correspond to the embedding V3E05L01N01T11 occupy.

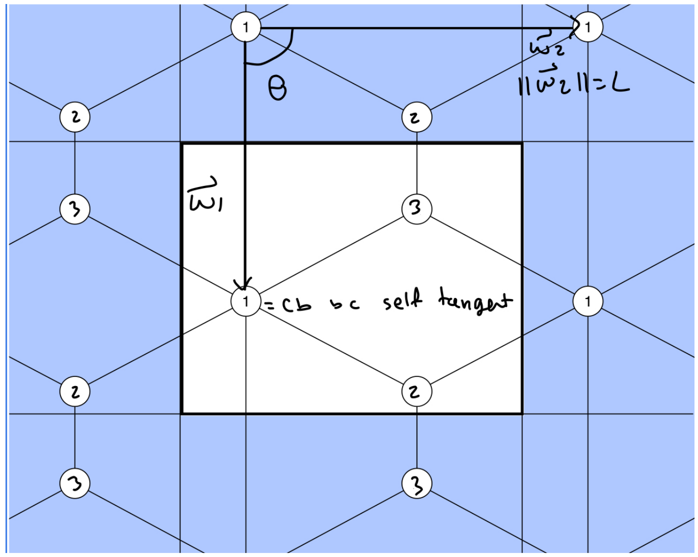


Figure 38: We choose θ to be the angles between $\vec{w_1}$ and $\vec{w_2}$ We note that V_1 must correspond to C_b and we let, without loss of generality, V_2 correspond to C_{s_1} and V_3 correspond to C_{s_2}.

by L. We note that V_1 must correspond to C_b as Propositions 3.4 and 3.3 show us that only C_b can self-tangent in our packing. Then, without loss of generality, we let V_2 correspond to C_{s_1} and V_3 correspond to C_{s_2}.

We lift the packing on T to the Euclidean plane. Using trivial translation of the plane we translate the center of C_b to $(0, 0)$. We then find the centers of the other circles and the endpoints of $\vec{w_1}$ and $\vec{w_2}$. We note that by Proposition 3.5, because C_b is self-tangent, r_b must be $\frac{1}{2}$.

We verify that these centers and endpoints of the lattice vectors in fact do give us the right embeddings on Mathematica by checking the distances between the centers of the lifts, i.e., the edge lengths in our embeddings. Since we have that $r_b = \frac{1}{2}$, all the edge lengths should be of magnitude $r_b + r_s = \frac{1}{\sqrt{2}}$, or $r_s + r_s = \sqrt{2} - 1$, or $r_b + r_b = 1$, where the magnitude depends on the tangencies.

Now we impose bounds on θ and L so that the bounded parameters with the centers and endpoints we found will give us the correct embedding graphs, locally maximally dense packings with no free circles, and will not over count any packings. Figure 39 shows what a general locally maximally dense packing that corresponds to V3E05L01N01T31 looks like. We verify that these packings are in fact locally maximally dense with no freee circles by checking if the packings are properly stressed. See Mathematica document for the stresses.

Now we determine what region of the moduli space the locally maximally dense packings with no free circles that correspond to V3E05L01N01T11 occupy. We get the region depicted in Figure 40.

We note that, as we said previously, the locally maximally dense packings with no free circles that correspond to the embedding V3E05L01N01T31 occupy a closed one dimensional region in the moduli space, bordering part of the open two dimensional region that the locally maximally dense packings with no free circles that correspond to the embedding V3E05L00N01T21 occupy.

In this next subsection we analyze the last other embedding with locally maximally dense packings with no free

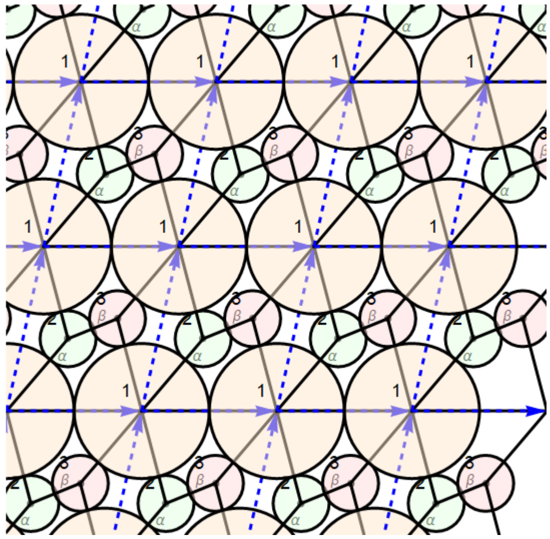


Figure 39: This figure shows what a typical locally maximally dense packing that corresponds to V3E05L01N01T31 looks like.

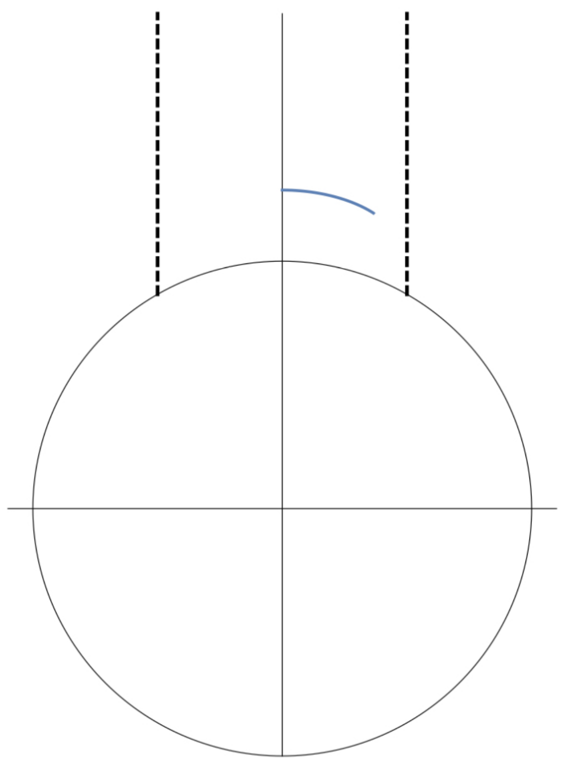


Figure 40: The one dimensional blue region is the region of the moduli space that the locally maximally dense packings with no free circles that correspond to the embedding V3E05L01N01T31 occupy.

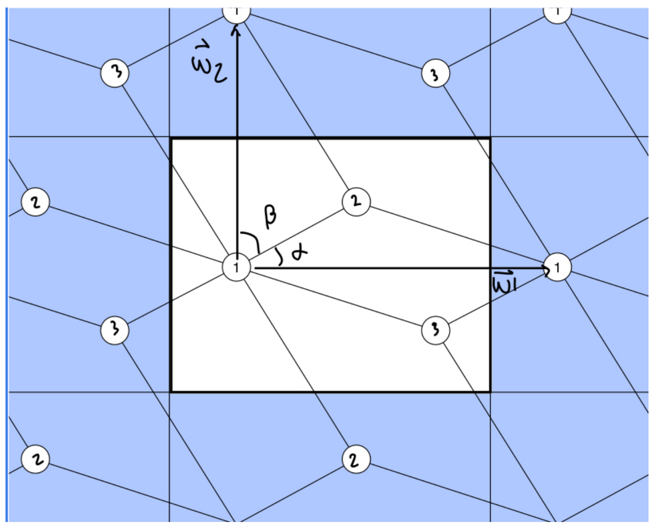


Figure 41: We choose α and β to be the angles between the segments $\overline{V_1V_2}$ and $\vec{w_1}$, and $\overline{V_1V_2}$ and $\vec{w_2}$, respectively. We note that V_1 must correspond to C_b and we let, without loss of generality, V_2 correspond to C_{s_1} and V_3 correspond to C_{s_2}.

circles that occupy to an open two dimensional region in the moduli space.

6.6 V3E06L00N02T11

We now analyze the last embedding which has locally maximally dense packings with no free cricles that occupy an open two dimensional region in the moduli space: V3E06L00N02T11, as shown in Figure 41. We first choose α and β to be the angles between the segments $\overline{V_1V_2}$ and $\vec{w_1}$, and $\overline{V_1V_2}$ and $\vec{w_2}$, respectively, as pictured in Figure 41. Our choice of $\vec{w_1}$ and $\vec{w_2}$ is arbitrary. We note that V_1 must correspond to C_b as Propositions 3.2 and 3.9 show us that only C_b can be 2-tangent to two other circles in our packing. Then, without loss of generality, we let V_2 correspond to C_{s_1} and V_3 correspond to C_{s_2}.

We lift the packing on T to the Euclidean plane. Using trivial translation of the plane we translate the center of C_b to $(0, 0)$. We then find the centers of the other circles and the endpoints of $\vec{w_1}$ and $\vec{w_2}$. We scale the whole packing so that $r_b = \frac{1}{2}$.

We verify that these centers and endpoints of the lattice vectors in fact do give us the right embeddings on Mathematica by checking the distances between the centers of the lifts, i.e., the edge lengths in our embeddings. Since we chose to scale r_b to $\frac{1}{2}$, all the edge lengths should be of magnitude $r_b + r_s = \frac{1}{\sqrt{2}}$, or $r_s + r_s = \sqrt{2} - 1$, or $r_b + r_b = 1$, where the magnitude depends on the tangencies.

Now we impose bounds on α and β so that the bounded parameters with the centers and endpoints we found will give us the correct embedding graphs, locally maximally dense packings with no free circles, and will not over count any packings. Figure 42 shows what a general locally maximally dense packing that corresponds to V3E05L00N01T11 looks like. We verify that these packings are in fact locally maximally dense with no freee circles by checking if the packings are properly stressed. See Mathematica document for the stresses.

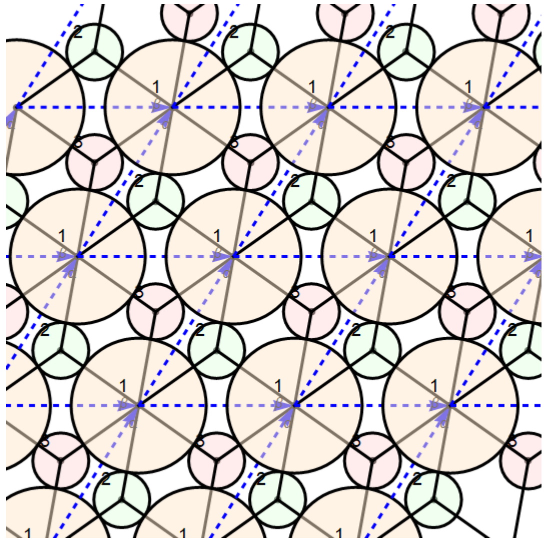


Figure 42: This figure shows what a typical locally maximally dense packing that corresponds to V3E06L00N02T11 looks like.

Now we determine what region of the moduli space the locally maximally dense packings with no free circles that correspond to V3E06L00N02T11 occupy. We get the region depicted in Figure 43.

We note that the locally maximally dense packings with no free circles that correspond to the embedding occupy an open two dimensional region in the moduli space. The locally maximally dense packings with no free circles that correspond to the embeddings V3E06L01N01T11, V3E07L00N01T11 and V3E07L01N01T11 border a part of this two dimensional region. V3E06L01N01T11 adds a self-tangency to V3E06L00N02T11; V3E07L00N01T11 adds another edge to V3E07L00N01T11; V3E07L01N01T11 adds a self-tangency and another edge to V3E07L01N01T11. We will further analyze V3E06L01N01T11, V3E07L00N01T11 and V3E07L01N01T11 in the following subsections.

6.7 V3E06L01N01T11

We now analyze the embedding V3E06L01N01T11, as shown in Figure 44. We first choose α and β to be the angles between the line segments $\overline{V_1V_2}$ and $\vec{w_1}$, and $\overline{V_1V_2}$ and $\vec{w_2}$ respectively, as pictured in Figure 44. We note that V_1 must correspond to C_b as Propositions 3.4 and 3.3 show us that only C_b can self-tangent in our packing. Then, without loss of generality, we let V_2 correspond to C_{s_1} and V_3 correspond to C_{s_2}.

We lift the packing on T to the Euclidean plane. Using trivial translation of the plane we translate the center of C_b to $(0,0)$. We then find the centers of the other circles and the endpoints of $\vec{w_1}$ and $\vec{w_2}$. We note that by Proposition 3.5, because C_b is self-tangent, r_b must be $\frac{1}{2}$.

We verify that these centers and endpoints of the lattice vectors in fact do give us the right embeddings on Mathematica by checking the distances between the centers of the lifts, i.e., the edge lengths in our embeddings. Since we have that $r_b = \frac{1}{2}$, all the edge lengths should be of magnitude $r_b + r_s = \frac{1}{\sqrt{2}}$, or $r_s + r_s = \sqrt{2} - 1$, or $r_b + r_b = 1$, where the magnitude depends on the tangencies.

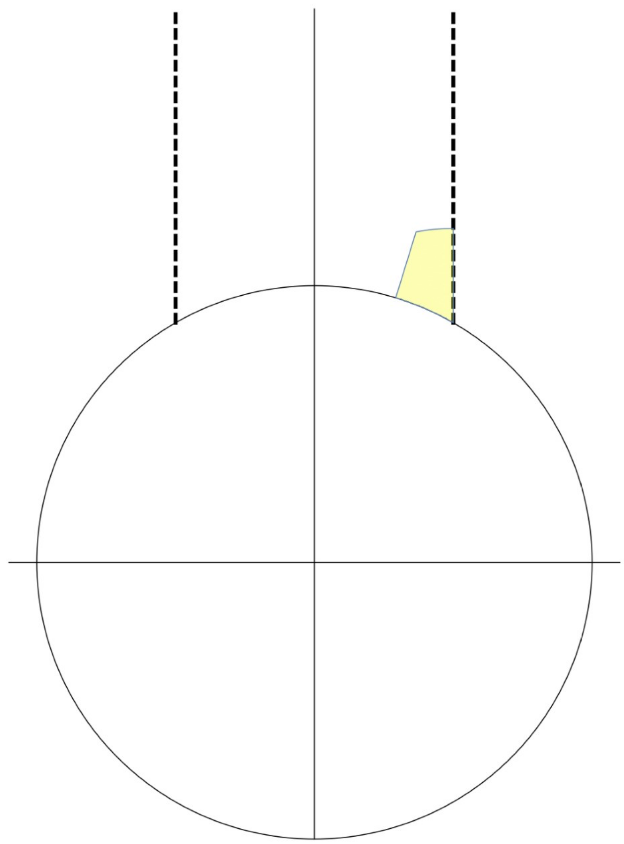


Figure 43: This is the region of the moduli space that the locally maximally dense packings with no free circles that correspond to the embedding V3E06L00N02T11 occupy.

Remark. *This yellow region actually consists of tori on which there exists two non homotopic locally maximally dense packings with no free circles that correspond to the embedding V3E06L00N02T11.*

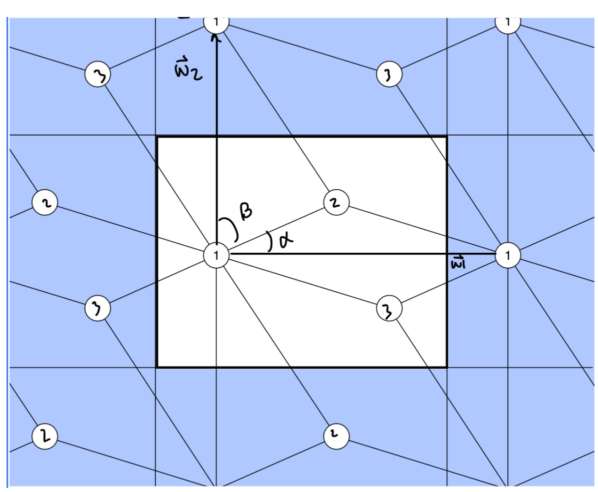


Figure 44: We choose α and β to be the angles between line segments $\overline{V_1V_2}$ and $\vec{w_1}$, and $\overline{V_1V_2}$ and $\vec{w_2}$ respectively. We note that V_1 must correspond to C_b and we let, without loss of generality, V_2 correspond to C_{s_1} and V_3 correspond to C_{s_2}.

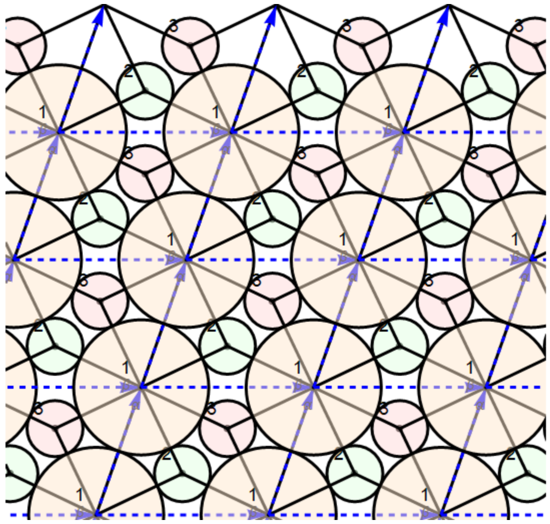


Figure 45: This figure shows what a typical locally maximally dense packing that corresponds to V3E06L01N01T11 looks like.

Now we impose bounds on α and β so that the bounded parameters with the centers and endpoints we found will give us the correct embedding graphs, locally maximally dense packings with no free circles, and will not over count any packings. Figure 45 shows what a general locally maximally dense packing that corresponds to V3E05L01N01T21 looks like. We verify that these packings are in fact locally maximally dense with no freee circles by checking if the packings are properly stressed. See Mathematica document for the stresses.

Now we determine what region of the moduli space the locally maximally dense packings with no free circles that correspond to V3E06L01N01T11 occupy. We get the region depicted in Figure 46.

We note that, as we said previously, the locally maximally dense packings with no free circles that correspond to the embedding V3E06L01N01T11 occupy a closed one dimensional region in the moduli space, bordering part of the open two dimensional region that the locally maximally dense packings with no free circles that correspond to the embedding V3E06L00N02T11. In the next subsection we analyze one of the remaining two embeddings that does this as well, V3E07L00N01T11.

6.8 V3E07L00N01T11

We now analyze the embedding V3E07L00N01T11, as shown in Figure 47. We first choose α and β to be the angles between the line segments $\overline{V_1V_2}$ and $\vec{w_1}$, and $\overline{V_1V_2}$ and $\vec{w_2}$ respectively, as pictured in Figure 47. We note that V_1 must correspond to C_b as Propositions 3.2 and 3.9 show us that only C_b can be 2-tangent to two other circles in our packing. Then, without loss of generality, we let V_2 correspond to C_{s_1} and V_3 correspond to C_{s_2}.

We lift the packing on T to the Euclidean plane. Using trivial translation of the plane we translate the center of C_b to $(0, 0)$. We then find the centers of the other circles and the endpoints of $\vec{w_1}$ and $\vec{w_2}$. We scale the whole packing so that $r_b = \frac{1}{2}$.

We verify that these centers and endpoints of the lattice vectors in fact do give us the right embeddings on

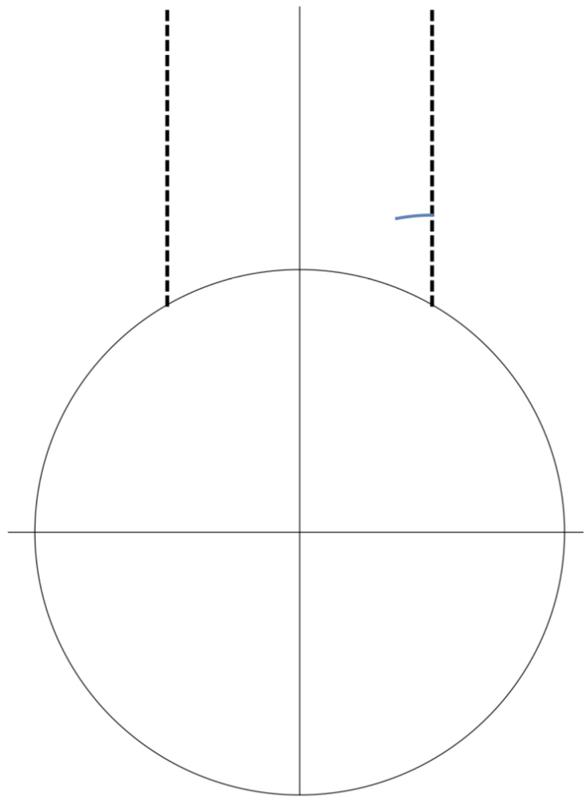


Figure 46: The one dimensional blue region is the region of the moduli space that the locally maximally dense packings with no free circles that correspond to the embedding V3E06L01N01T11 occupy.

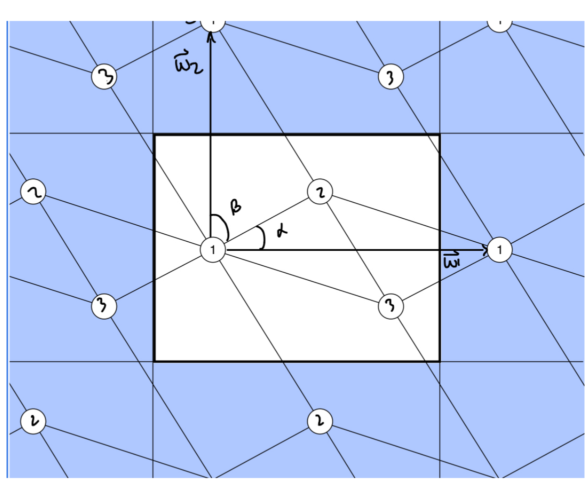


Figure 47: We choose α and β to be the angles between line segments $\overline{V_1V_2}$ and $\vec{w_1}$, and $\overline{V_1V_2}$ and $\vec{w_2}$ respectively. We note that V_1 must correspond to C_b and we let, without loss of generality, V_2 correspond to C_{s_1} and V_3 correspond to C_{s_2}.

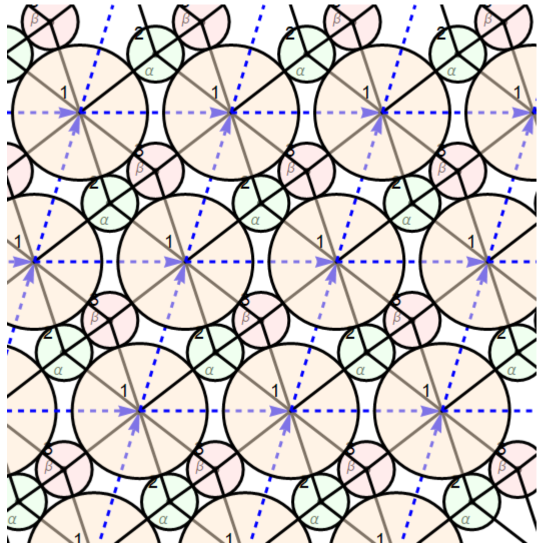


Figure 48: This figure shows what a typical locally maximally dense packing that corresponds to V3E07L00N01T11 looks like.

Mathematica by checking the distances between the centers of the lifts, i.e., the edge lengths in our embeddings. Since we have that $r_b = \frac{1}{2}$, all the edge lengths should be of magnitude $r_b + r_s = \frac{1}{\sqrt{2}}$, or $r_s + r_s = \sqrt{2} - 1$, or $r_b + r_b = 1$, where the magnitude depends on the tangencies.

Now we impose bounds on α and β so that the bounded parameters with the centers and endpoints we found will give us the correct embedding graphs, locally maximally dense packings with no free circles, and will not over count any packings. Figure 48 shows what a general locally maximally dense packing that corresponds to V3E07L00N01T11 looks like. We verify that these packings are in fact locally maximally dense with no freee circles by checking if the packings are properly stressed. See Mathematica document for the stresses.

Now we determine what region of the moduli space the locally maximally dense packings with no free circles that correspond to V3E07L00N01T11 occupy. We get the region depicted in Figure 52.

We note that, as we said previously, the locally maximally dense packings with no free circles that correspond to the embedding V3E07L00N01T11 occupy a closed one dimensional region in the moduli space, bordering part of the open two dimensional region that the locally maximally dense packings with no free circles that correspond to the embedding V3E06L00N02T11. In the next subsection we analyze the remaining embedding that does this as well, V3E07L01N01T11, but with a point instead of a one dimensional closed region.

6.9 V3E07L01N01T11

We analyze the last of our nine embeddings here, V3E07L01N01T11. We recall from Section 3 that V3E07L01N01T11 is described by the conditions of Proposition 3.10. That is, the locally maximally dense packings of V3E07L01N01T11 with no free circles correspond to at most one point in the moduli space. We verify this on Mathematica following our usual procedure.

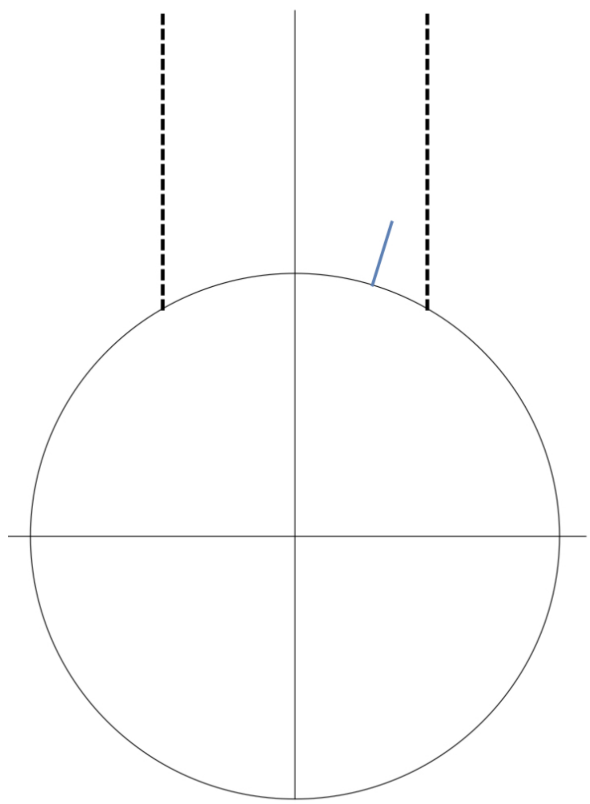


Figure 49: The one dimensional blue region is the region of the moduli space that the locally maximally dense packings with no free circles that correspond to the embedding V3E07L00N01T11 occupy.

Figure 50: We choose α to be the angle between $\vec{w_1}$ and $\vec{w_2}$. We note that V_1 must correspond to C_b and we let, without loss of generality, V_2 correspond to C_{s_1} and V_3 correspond to C_{s_2}.

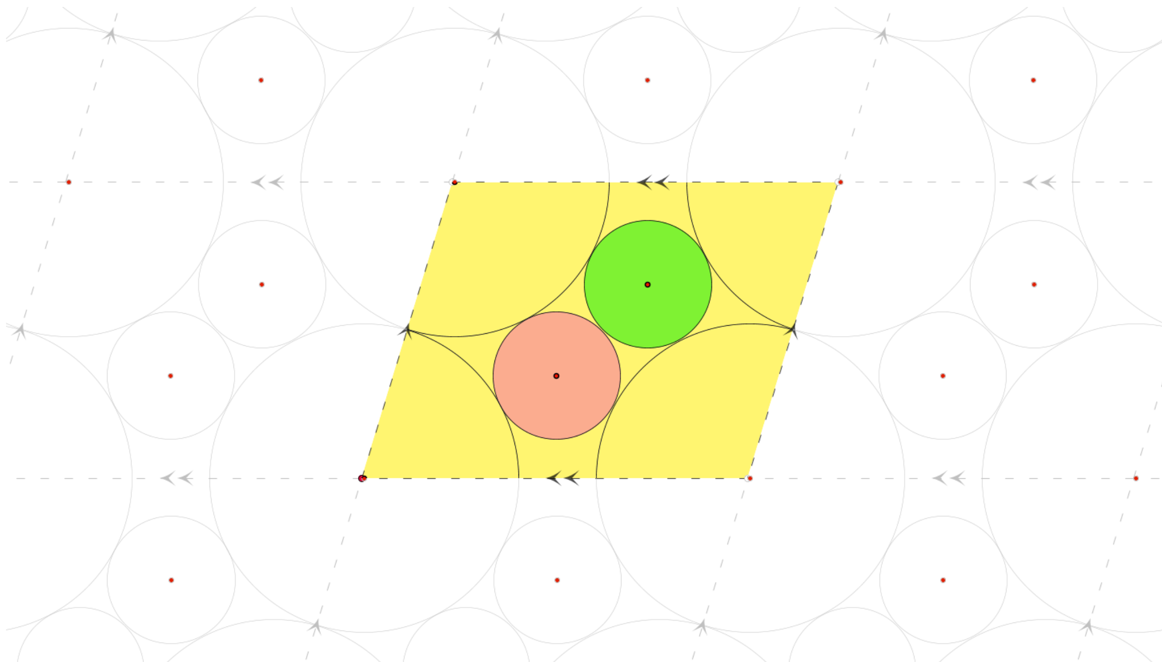


Figure 51: This figure shows what the locally maximally dense packing that corresponds to V3E07L00N01T11 looks like.

We first choose α to be the angle between $\vec{w_1}$ and $\vec{w_2}$, as pictured in Figure 50. We note that V_1 must correspond to C_b as Propositions 3.4 and 3.3 show us that only C_b can self-tangent in our packing. Then, without loss of generality, we let V_2 correspond to C_{s_1} and V_3 correspond to C_{s_2}.

We lift the packing on T to the Euclidean plane. Using trivial translation of the plane we translate the center of C_b to $(0, 0)$. We then find the centers of the other circles and the endpoints of $\vec{w_1}$ and $\vec{w_2}$. We note that by Proposition 3.5, because C_b is self-tangent, r_b must be $\frac{1}{2}$.

We verify that these centers and endpoints of the lattice vectors in fact do give us the right embeddings on Mathematica by checking the distances between the centers of the lifts, i.e., the edge lengths in our embeddings. Since we have that $r_b = \frac{1}{2}$, all the edge lengths should be of magnitude $r_b + r_s = \frac{1}{\sqrt{2}}$, or $r_s + r_s = \sqrt{2} - 1$, or $r_b + r_b = 1$, where the magnitude depends on the tangencies.

Now we impose bounds on α so that the bounds on α with the centers and endpoints we found will give us the correct embedding graphs, locally maximally dense packings with no free circles, and will not over count any packings. Figure 51 shows what the locally maximally dense packing that corresponds to V3E07L00N01T11 looks like. This result agrees with the results of Proposition 3.10 in Section 3. We verify that this packing is in fact locally maximally dense with no free circles by checking if the packing is properly stressed. See the Mathematica document.

Now we determine what region of the moduli space the locally maximally dense packing with no free circles that

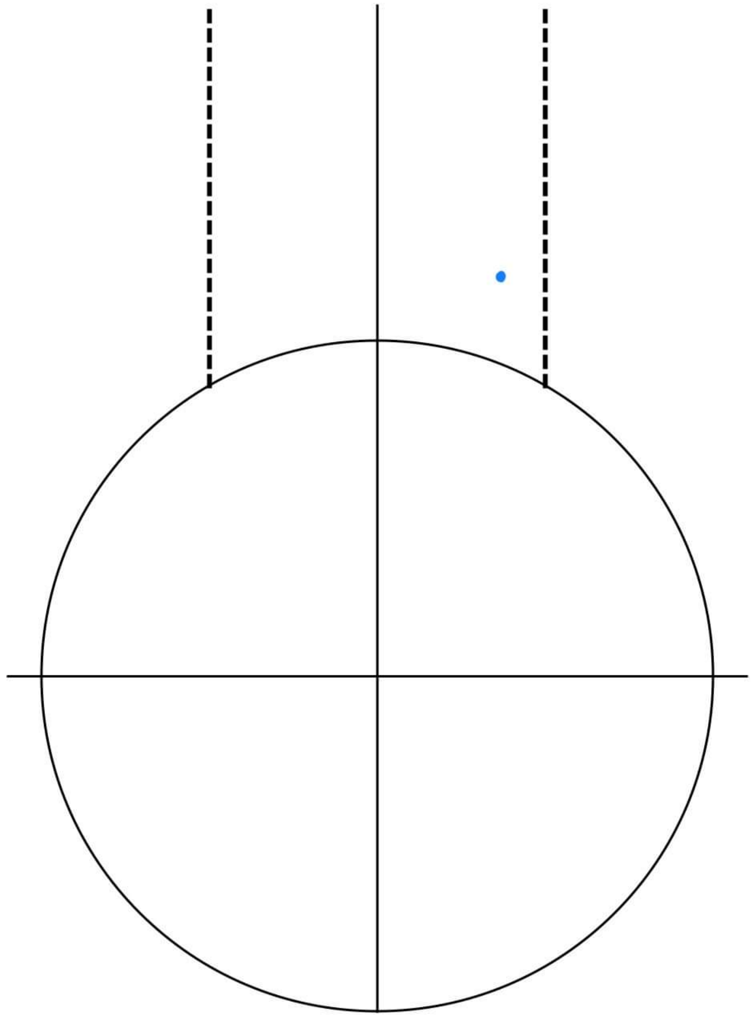


Figure 52: The blue point is the region of the moduli space that the locally maximally dense packing with no free circles that corresponds to the embedding V3E07L00N01T11 occupies.

corresponds to V3E07L00N01T11 occupies. We get the point depicted in Figure 52.

We note that, as we said previously, the locally maximally dense packing with no free circles that corresponds to the embedding V3E07L00N01T11 corresponds to a point in the moduli space, on the boundary of the open two dimensional region that the locally maximally dense packings with no free circles that correspond to the embedding V3E06L00N02T11 occupy.

We are now done with analyzing the remaining nine embeddings separately, and we have the regions of the moduli space that the locally maximally dense packings with no free circles, of circles C_b, C_{s_1}, C_{s_2}, with a radius ratio of $\frac{r_s}{r_b} = \sqrt{2} - 1$, occupy. We can summarize the results of this section in Figure 53, which shows the covering of the moduli space by the regions that the locally maximally dense packings with no free cricles that correspond to the remaining nine embeddings, occupy. There exists overlaps in the moduli space of the regions that the locally maximally dense packings with no free cricles that correspond to *different* embeddings, occupy. This signifies that

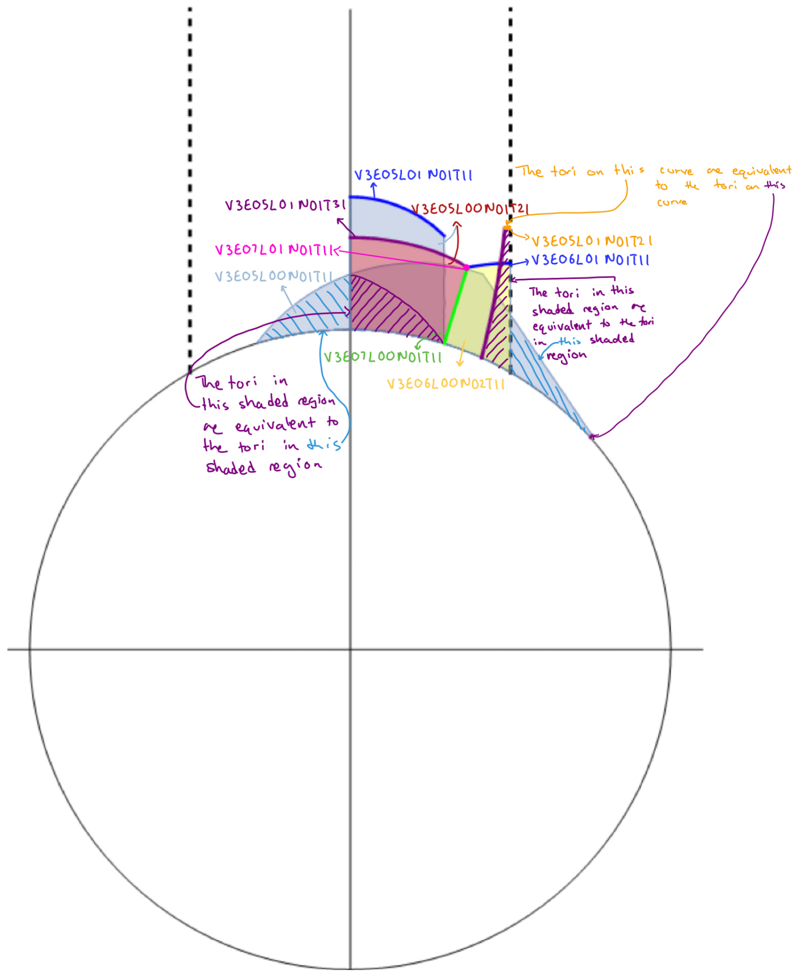


Figure 53: This figure shows the division of the moduli space into the regions that the locally maximally dense packings with no free cricles that correspond to the remaining nine embeddings, occupy. Will- I will fix this picture as well with the correct region for V3E05L00N01T11 once I get the picture from Dan.

there exsits multiple locally maximally dense packing with no free circles on the tori defined by the regions in the moduli space with the overlaps.

In the next section, we will compare the denisities of all locally maximally dense packings with no free circles on all flat tori to get the globally maximally dense packings with no free circles on all flat tori.

7 Main Results: Locally and Globally Maximally Dense Packings of C_b, C_{s_1} and C_{s_2} on All Flat Tori

We start this section by classifying the regions of the moduli space by the packings that will be locally maximally dense in those regions.

Theorem 7.1. *All locally maximally dense packings of circles C_b, C_{s_1}, C_{s_2} with radius ratio $\frac{r_s}{r_b}$ on any flat torus can be classified as shown in Figure 54.*

- *In region A of the moduli space, the blue curve, we have that the locally maximally dense packings of C_b, C_{s_1}, C_{s_2} with radius ratio $\frac{r_s}{r_b}$ correspond to the free circle packing, and the packings that correspond to the embedding graph of* V3E05L01N01T11.

- In region B of the moduli space, the light blue open two dimensional region, we have that the locally maximally dense packings of C_b, C_{s_1}, C_{s_2} with radius ratio $\frac{r_s}{r_b}$ correspond to the free circle packing, and the packings that correspond to the embedding graph of V3E05L00N01T21.

- In region C of the moduli space, the purple curve, we have that the locally maximally dense packings of C_b, C_{s_1}, C_{s_2} with radius ratio $\frac{r_s}{r_b}$ correspond to the packings that correspond to the embedding graph of V3E05L01N01T31.

- In region D of the moduli space, the top part of the open two dimensional shaded purple region, we have that the locally maximally dense packings of C_b, C_{s_1}, C_{s_2} with radius ratio $\frac{r_s}{r_b}$ correspond to the free circle packing, and the packings that correspond to the embedding graph of V3E05L00N01T11.

- In region E of the moduli space, the pink dot, we have that the locally maximally dense packings of C_b, C_{s_1}, C_{s_2} with radius ratio $\frac{r_s}{r_b}$ correspond to the packing that corresponds to the embedding graph of V3E07L01N01T11.

- In region F of the moduli space, the blue curve, we have that the locally maximally dense packings of C_b, C_{s_1}, C_{s_2} with radius ratio $\frac{r_s}{r_b}$ correspond to the packings that correspond to the embedding graph of V3E06L01N01T11.

- In region G of the moduli space, overlayed by the two dimensional open red region, we have that the locally maximally dense packings of C_b, C_{s_1}, C_{s_2} with radius ratio $\frac{r_s}{r_b}$ correspond to the packings that correspond to the embedding graphs of V3E05L00N01T21 and V3E05L00N01T11.

- In region H of the moduli space, the green curve, we have that the locally maximally dense packings of C_b, C_{s_1}, C_{s_2} with radius ratio $\frac{r_s}{r_b}$ correspond to the packings that correspond to the embedding graph of V3E07L00N01T11.

- In region I of the moduli space, overlayed by the two dimensional open yellow region, we have that the locally maximally dense packings of C_b, C_{s_1}, C_{s_2} with radius ratio $\frac{r_s}{r_b}$ correspond to the packings that correspond to the embedding graphs of V3E06L00N02T11 and V3E05L00N01T11.

- In region J of the moduli space, the purple curve, we have that the locally maximally dense packings of C_b, C_{s_1}, C_{s_2} with radius ratio $\frac{r_s}{r_b}$ correspond to the packings that correspond to the embedding graph of V3E05L01N01T21.

- In region K of the moduli space, the white region, we have that the locally maximally dense packings of C_b, C_{s_1}, C_{s_2} with radius ratio $\frac{r_s}{r_b}$ are those in which all circles are free.

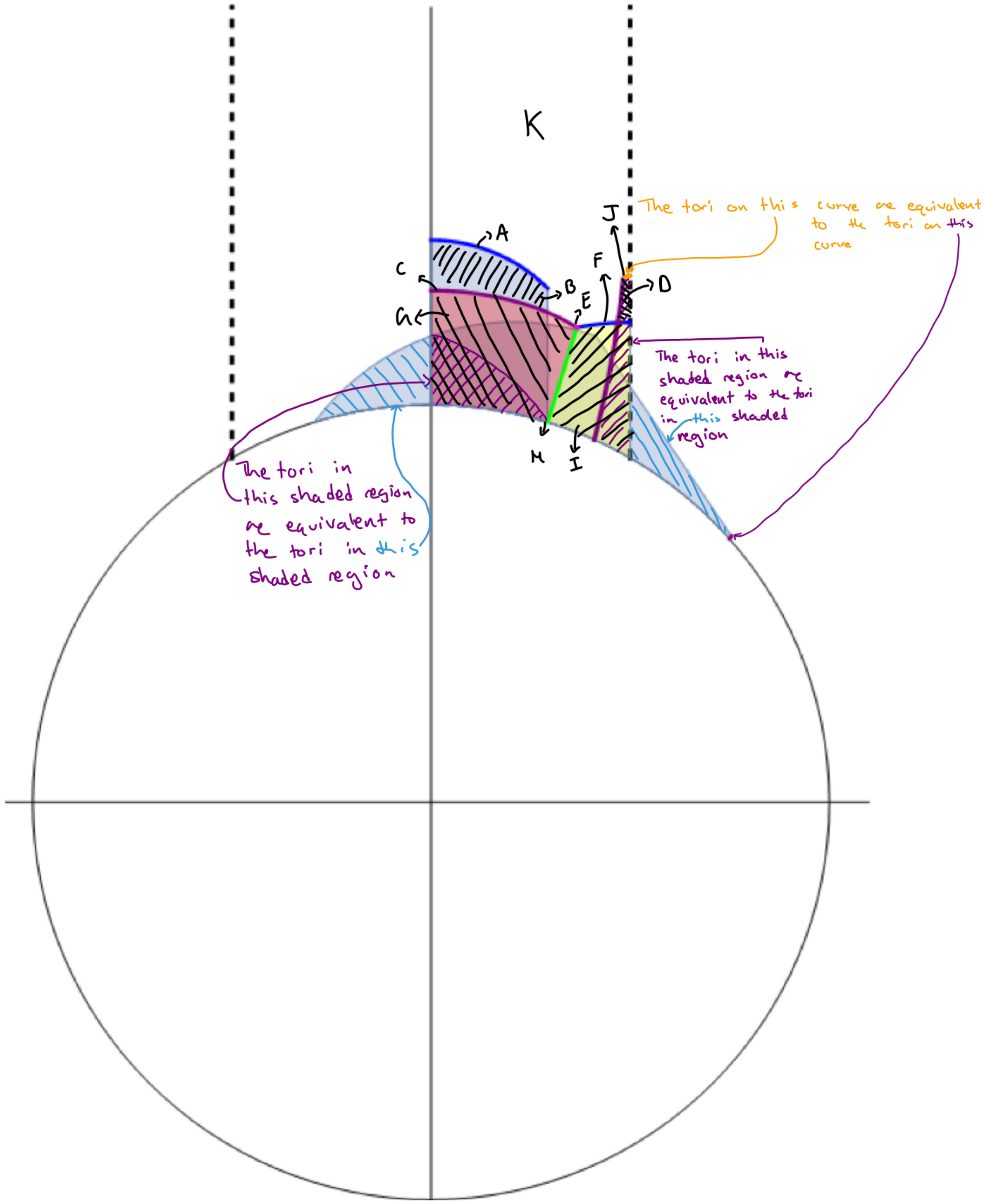


Figure 54: All locally maximally dense packings with no free cirlces of circles C_b, C_{s_1}, C_{s_2} with radius ratio $\frac{r_s}{r_b}$ on any flat torus can be classified as such. Will- I will fix this picture as well with the correct region for V3E05L00N01T11 once I get the picture from Dan.

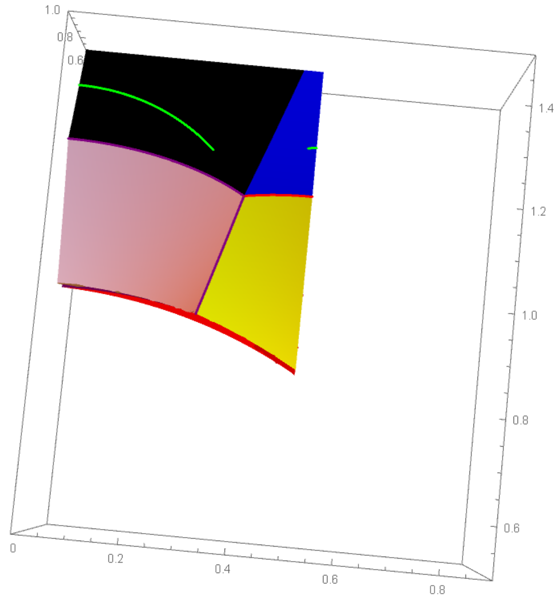


Figure 55: This is the top down view of the density plot of the regions mentioned in Theorem 7.1.

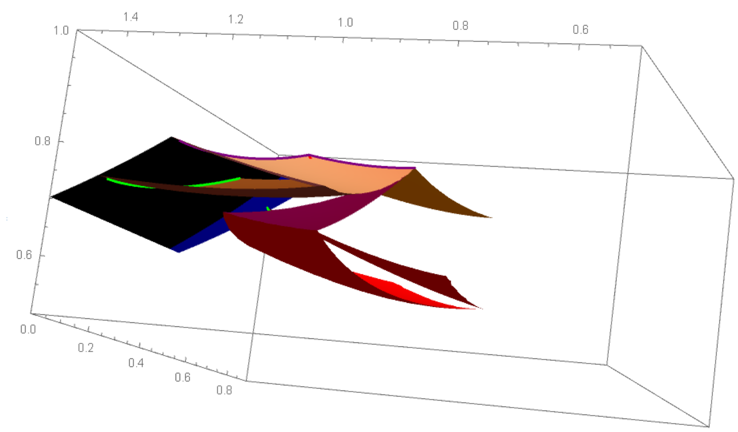


Figure 56: This is a side view of the density plot of the regions mentioned in Theorem 7.1.

Remark. *The reason we don't have the free circle packing as a locally maximally dense packing in every point in the moduli space is because the tori at those points can not allow free circles without overlaps. That is, those tori can not "fit" in free circles.*

Now we compare the densities of the regions mentioned in Theorem 7.1. We find the densities of the regions in Theorem 7.1 on Mathematica. We get a three dimensional plot, which is depicted in Figures 55, 56, 57. The density plot shows the regions mentioned in Theorem 7.1 and the densities across those regions. It should be noted that the black and dark blue regions in the density plot correspond to the free circle packings.

Figure 55 shows us the regions mentioned in Theorem 7.1 with the greatest densities. This is because if a region is below another in the density plot, that is because it has a lower density. Therefore, we have the following result.

Theorem 7.2. *All globally maximally dense packings of circles C_b, C_{s_1}, C_{s_2} with radius ratio $\frac{r_s}{r_b}$ on any flat torus*

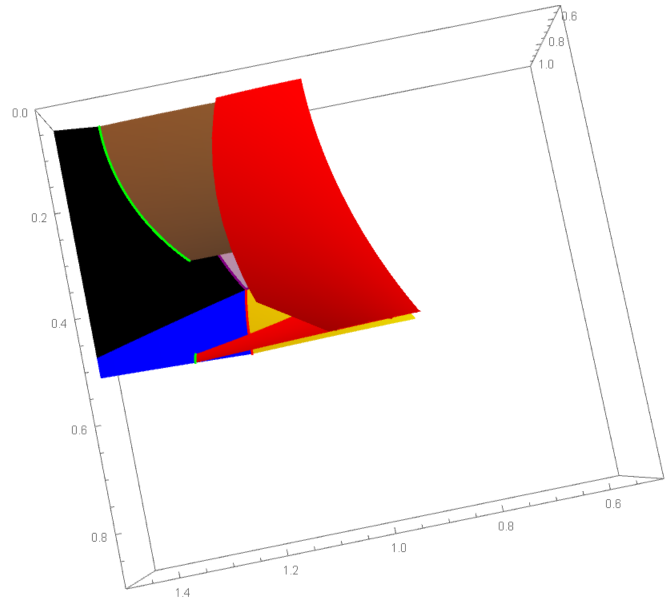


Figure 57: This is the bottom up view of the density plot of the regions mentioned in Theorem 7.1.

can be classified as shown in Figure 58

- *In region A of the moduli space, the blue curve, we have that the globally maximally dense packings of* C_b, C_{s_1}, C_{s_2} *with radius ratio* $\frac{r_s}{r_b}$ *corresponds to the packings that correspond to the embedding graph of* V3E05L01N01T11.

- In region C of the moduli space, the purple curve, we have that the globally maximally dense packings of C_b, C_{s_1}, C_{s_2} with radius ratio $\frac{r_s}{r_b}$ correspond to the packings that correspond to the embedding graph of V3E05L01N01T31.

- In region E of the moduli space, the pink dot, we have that the globally maximally dense packings of C_b, C_{s_1}, C_{s_2} with radius ratio $\frac{r_s}{r_b}$ correspond to the packing that corresponds to the embedding graph of V3E07L01N01T11.

- In region F of the moduli space, the blue curve, we have that the globally maximally dense packings of C_b, C_{s_1}, C_{s_2} with radius ratio $\frac{r_s}{r_b}$ correspond to the packings that correspond to the embedding graph of V3E06L01N01T11.

- In region G of the moduli space, the two dimensional open pink region, we have that the globally maximally dense packings of C_b, C_{s_1}, C_{s_2} with radius ratio $\frac{r_s}{r_b}$ correspond to the packings that correspond to the embedding graphs of V3E05L00N01T21.

- In region H of the moduli space, the green curve, we have that the globally maximally dense packings of C_b, C_{s_1}, C_{s_2} with radius ratio $\frac{r_s}{r_b}$ correspond to the packings that correspond to the embedding graph of V3E07L00N01T11.

- In region I of the moduli space, the two dimensional open yellow region, we have that the globally maximally dense packings of C_b, C_{s_1}, C_{s_2} with radius ratio $\frac{r_s}{r_b}$ correspond to the packings that correspond to the embedding graphs of V3E06L00N02T11.

- In region J of the moduli space, the purple curve, we have that the globally maximally dense packings of C_b, C_{s_1}, C_{s_2} with radius ratio $\frac{r_s}{r_b}$ correspond to the packings that correspond to the embedding graph of V3E05L01N01T21.

- In region K of the moduli space, the white region, we have that the globally maximally dense packings of C_b, C_{s_1}, C_{s_2} with radius ratio $\frac{r_s}{r_b}$ are those in which all circles are free.

The denisities of just the regions mentioned in Theorem 7.2 can be depicted using a contour plot. See Figure 59

Corollary. *The packing depicted in Figure 60 has the highest density of all packings of circles C_b, C_{s_1}, C_{s_2} with radius ratio $\frac{r_s}{r_b}$ on any flat torus.*

We have thus achieved our goal of finding the locally and globally maximally dense packings of circles C_b, C_{s_1}, C_{s_2} with radius ratio $\frac{r_s}{r_b}$ on any flat torus.

References

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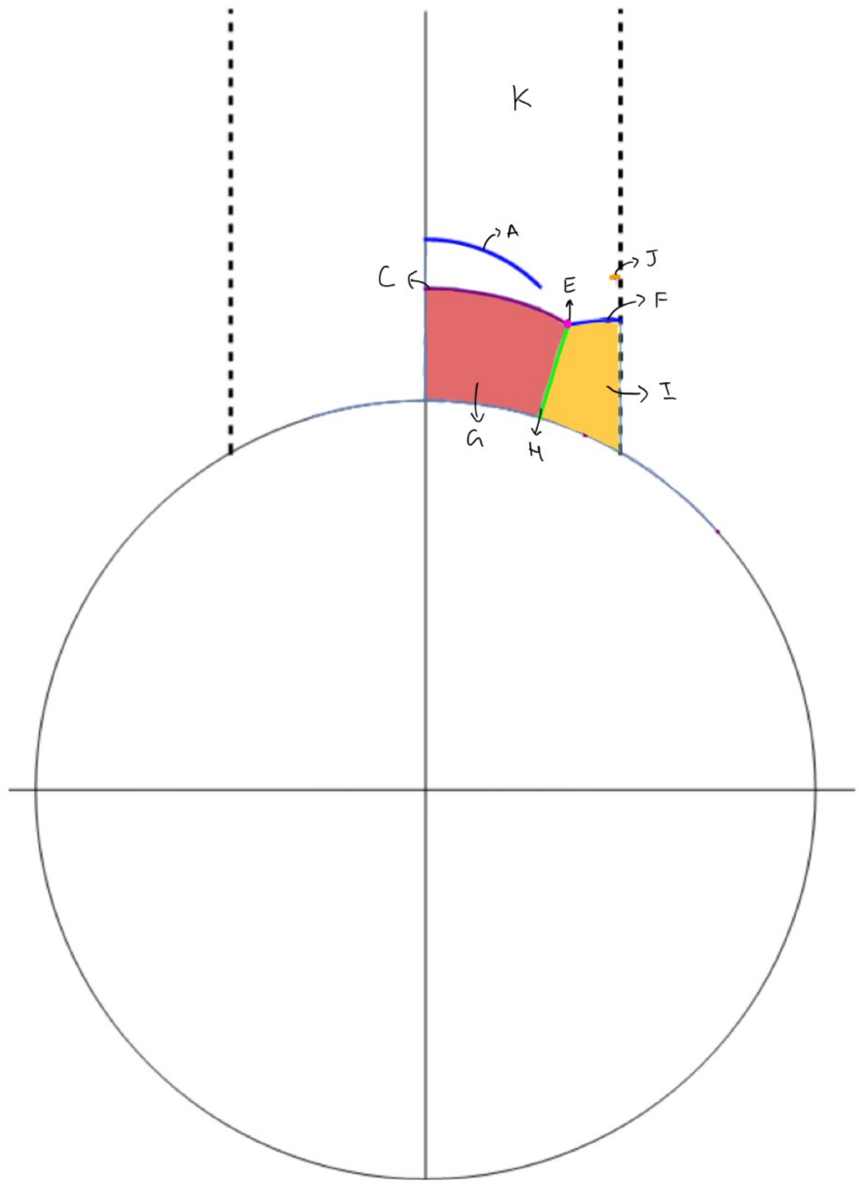

Figure 58: All globally maximally dense packings with no free cirlces of circles C_b, C_{s_1}, C_{s_2} with radius ratio $\frac{r_s}{r_b}$ on any flat torus can be classified as such.

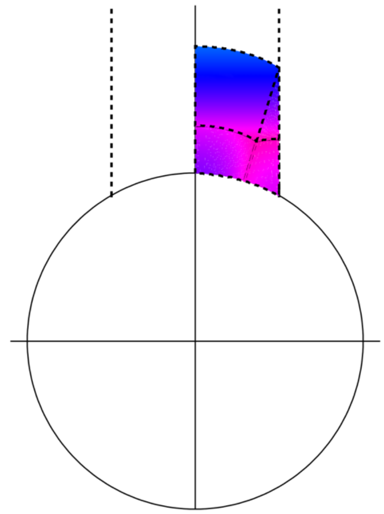

Figure 59: This contour plot depicts the densities of the regions mentioned in 7.2. The pinker region have higher densities than the bluer regions. The purplish regions have higher densities than the bluer regions, but lower densities than the pinker regions.

Remark. *While the contour plot seems to end at a certain curve above the blue region, that is not the case. The density steadily decreases after that curve as the area of the tori steadily increases, while the radii of our circles stay constant.*

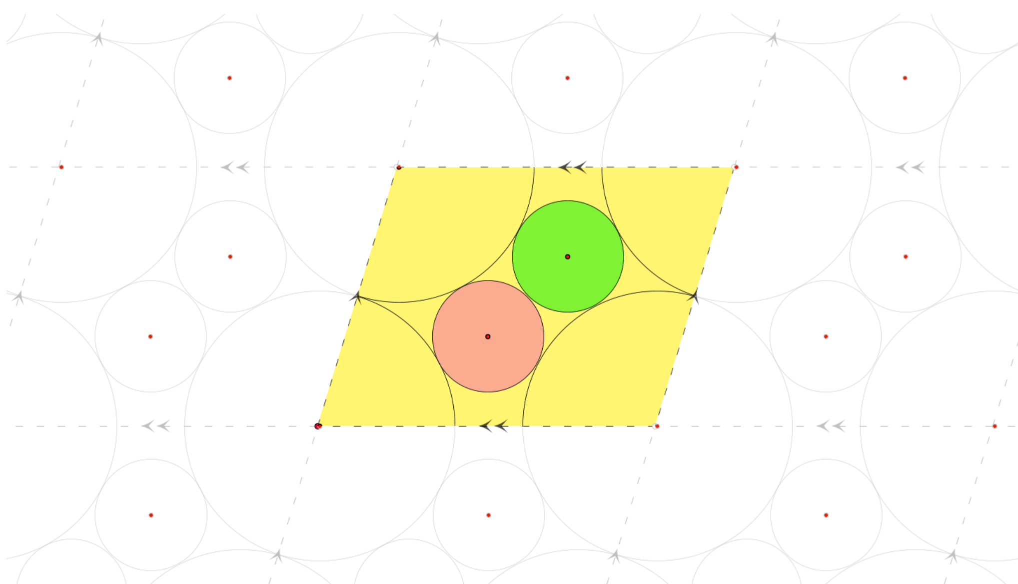

Figure 60: This packing has the highest density of all packings of circles C_b, C_{s_1}, C_{s_2} with radius ratio $\frac{r_s}{r_b}$ on any flat torus.